CHAPTER I INTRODUCTION & PRELIMINARIES

1.1 INTRODUCTION

In 1965, O.Njastad [44] defined and studied the concept of α -sets. Later, these are called as α-open sets in topology. In 1983, A. S. Mashhour et al [32], have defined the concepts like α-closed sets, α -closure of a set, α-openness and α - closedness in topology. Later, many topologists have studied these concepts in detail. G.B. Navalagi [35] has given some properties on α- neighbourhoods. In this paper we give some more properties of α-open sets.

Throughout this paper, (X, τ), (Y,σ), and so forth (or simply X, Y, etc.) will always denote topological spaces.

1.2 PRELIMINARIES

DEFINITION 1.2.1: Let X be a topological space and $A \subset X$. Then A is called

a) an **α-open** [44] if $A \subset int(cl(Int(A)))$

b) a **Semi-open** [24] if $A \subset cl(Int(A))$

c) a **Pre-open** [33] if $A \subset int(cl(A))$

d) a **Regular open** [28] if $A = int\ cl(A)$

The family of all α–open (resp. semi-open, pre-open) sets in a space X is denoted by $\alpha O(X)$ or τ^{α} (resp. SO(X) PO(X.) The complement of an α-open (resp. semi-open, pre-open) set is said to be α-

closed (resp. semi-closed, pre-closed), denoted by $\alpha F(X)$ (resp. SF(X) PF(X.).

Clearly, every open set is α -open. Also every closed set is α -closed (resp. semi-closed, pre-closed).

But converse need not be true.

DEFINITION 1.2.2 :Let X be a topological space and $A \subset X$. Then A is called a **α- neighbourhood** [35], denoted by α-nhd, of a point x in X, if there exist a α-open set U in X such that $x \in U \subset A$. The α-nhd system of a point $x \in X$ is denoted by α-N(x).

NOTE 1.2.3: Clearly every α -open set of A is α -nhd of the points of A.

DEFINITION 1.2.4 : The union of all semi-open (preopen, α-open) sets contained in A is called the **semi-interior of A,** denoted by **sint(A)** [14] **(**resp**. pre interior of A, pIntA** [34] **, α-interior of A, I_α(A)** [32].

DEFINITION 1.2.5: The intersection of all semi-closed (preclosed and α-closed) sets containing A is called the **semi-closure of A** , denoted by **scl(A)** [14 & 11] **(**resp**.preclosure of A , pcl A** [18 & 34], **α-closure of A, C_α(A)** [35]**) .**

DEFINITION 1.2.6: The union of all α-open sets contained in the complement of A is called the **α-exterior** of A, denoted by $Ext_\alpha(A)$ [7]

DEFINITION 1.2.7 : A point $x \in X$ is said to be a **α-limit point** [35] of $A \subset X$ iff $U \in \alpha O(X)$ implies $U \cap (A-\{x\}) \neq \emptyset$.

DEFINITION 1.2.8 : The set of all α-limit points of $A \subset X$ is called the **α-derived** set [35] of A and is denoted by $D_\alpha(A)$.

DEFINITION 1.2.9 : The set C_α - I_α is called the **α -frontier** [35] of A is denoted by $F_\alpha(A)$.

DEFINITION 1.2.10: The set $b_\alpha = A - I_\alpha(A)$ [or $b_\alpha = A \backslash I_\alpha(A)$] is said to **α-Boundary of A** or **α-border of A** [7].

DEFINITION 1.2.11: A subset A of a space (X ,τ) is called a α -generalized closed (or **αg-closed**) [31] (generalized preclosed or **gp-closed** [3],) if $C_\alpha(A) \subset U$ (pclA $\subseteq$ U) whenever $A \subset U$ and U is an open set.

DEFINITION 1.2.12: A subset A of X is called a **αg –open** set (gp-open set) if its complement is αg-closed set (gp-closed set) in X.

The family of all αg-open sets (gp-open sets) in X is denoted by αgO(X) (GPO(X)) and the family of all αg -closed sets (gp-closed sets) of X is denoted by αgF(X) (GPC(X)).

DEFINITION 1.2.13 :Let X be a topological space and $A \subset X$. Then A is called a **αg- neighbourhood** , denoted by αg-nhd, of a point x in X, if there exist a αg-open set U in X such that $x \in U \subset A$. The αg-nhd system of a point $x \in X$ is denoted by αg-N(x).

DEFINITION 1.2.14: A point $x \in A$ is said to be an **αg-interior** point of A if A is αg-nhd of x. In other words, it means that there exists a αg-open set G containing x such that $G \subset A$

The set of all αg-interior points of A is said to be αg-interior of A and is denoted by $I_{\alpha g}(A)$.

DEFINITION 1.2.15:[42] A subset A of a topological space X is called **γ_g –open** set iff $A \cap B \in GPO(X)$ for every $B \in GPO(X)$.

The family of all γ_g –open subset of X is denoted by γ_g O(X). The complement of a γ_g –open set is called γ_g –closed set. The family of γ_g-closed subsets of X is denoted by γ_gC(X).

DEFINITION 1.2.16: [19] A subset A of a topological space X is called $\gamma_{g\alpha g}$ -open set iff $A \cap B \in GPO(X)$ for every $B \in \alpha gO(X)$.

The family of all $\gamma_{g\alpha g}$-open subset of X is denoted by $\gamma_{g\alpha g}$O(X). The complement of a $\gamma_{g\alpha g}$-open set is called $\gamma_{g\alpha g}$-closed set. The family of $\gamma_{g\alpha g}$- closed subsets of X is denoted by $\gamma_{g\alpha g}$C(X).

DEFINITION 1.2.17: A function f: $X \rightarrow Y$ is said to be **α-continuous** [32] (resp. **pre-continuous** [33], **semi-continuous**[24]) if $f^{-1}(V)$ is α-open (resp. pre-open, semi-open) in X for every open set V of Y.

DEFINITION 1.2.18: A function f: $X \rightarrow Y$ is said to be **strongly pre-continuous** [33] (**strongly semi-continuous** [1] if $f^{-1}(V)$ is open in X for every pre-open (semi-open) set V of Y.

DEFINITION 1.2.19: A function f: $X \rightarrow Y$ is said to be **preirresolute** [47] (resp. **irresolute** [13] , and **α-irresolute** [27]) if $f^{-1}(V)$ is preopen (resp. semiopen and α-open) set in X for each preopen (resp. semiopen and α-open) set of Y.

DEFINITION 1.2.20 : A function f :X $\rightarrow$ Y is said to be **strongly -irresolute**[16] if for each $x \in X$ and each $V \in SO(Y)$ containing f(x), there exists $U \in SO(X)$ containing x such that $f(sclU) \subseteq V$.

DEFINITION 1.2.21: A function f:X$\rightarrow$Y is said to be **contra α-continuous** [20] (resp. **contra pre-continuous** [22]) if $f^{-1}(V)$ is α –closed (resp. pre-closed) in X for every open set V of Y.

DEFINITION 1.2.22:A space X is said to be **α-T_0** [35]) if for each pair of distinct points in X, there exists a α-open set of X containing one point but not the other .

DEFINITION 1.2.23 : A space X is said to be **α -T_1** [35] if for each pair of distinct points x and y of X , there exist α-open sets U and V containing x and y resp. such that $y \notin U$ and $x \notin V$.

DEFINITION 1.2.24 : A space X is said to be **α-T_2** [35] if for each pair of distinct points x and y in X , there exist disjoint α-open sets U and V in X such that $x \in U$ and $y \in V$.

DEFINITION 1.2.25 [30]:A space X is said to be **αT_1** if (X, τ^{α}) is a T_1.

DEFINITION 1.2.26 [7] **:** A space X is said to be an **αT_b** if every αg-closed set is closed.

DEFINITION 1.2.27 : A function $f : X \rightarrow Y$ is said to be **preopen** [33] **(resp. semiopen** [5], **α-open** [32]) if f(U) is preopen (resp. semiopen , α-open) in Y for each open set U in X.

DEFINITION 1.2.28: A function $f : X \rightarrow Y$ is said to be **preclosed** [18] (**resp. semiclosed** [6] **,** **α-closed** [32]) if f(F) is preclosed (resp. semiclosed , α-closed) set in Y for each closed set F in X .

DEFINITION 1.2.29 [48] : A function $f:X\rightarrow Y$ is called (i) **strongly α-open** (**resp. strongly semiopen , strongly preopen**) if the image of each α-open (resp. semiopen , preopen) set in X is a α-open (resp. semiopen , preopen) set in Y .

CHAPTER 2 BASIC PROPERTIES OF α-OPEN SETS αg- OPEN SETS AND $\gamma_{g\alpha g}$ –OPEN SETS

2.1 INTRODUCTION

G.B.Navalagi[35], has given some properties on α-neighbourhoods. Latter many topologists have studied the properties of α- neighbourhoods of a point.

Levine[25] introduced the class of generalized closed sets (g-closed sets) and studied their most basic properties. The concepts of gα-closed and αg-closed sets were introduced and all these notions were defined through α-open sets. In this chapter we study some more properties α-neighbourhoods of a point, , α-interior of a set, α - closure and α- derived sets, αg neighbourhoods of a point, αg- interior of a set, αg - closure and αg- derived sets.

In 1987, Andrijevic [3] introduced a new class of sets called γ-open sets. In this chapter we introduce a new set which is analogous to **γ-**open sets but whose concept is defined via αg-open sets and GP-open sets. We shall call this new class of sets as $\gamma_{g\alpha g}$ –open sets.

2.2 SOME MORE PROPERTIES OF α- NEIGHBOURHOODS OF A POINT, α-INTERIOR OF A SET, α- CLOSURE AND α–DERIVATIVE SETS

LEMMA 2.2.1:[35] Any arbitrary union of α-nhds of a point x is again a α-nhd of x.

LEMMA 2.2.2: Finite intersection of α-nhds of a point x is again a α-nhd of x.

THEOREM 2.2.3: Every nhd of x in X is α-nhd of x.

PROOF: Easy to prove

Converse of the theorem 2.2.3 is not true in general, For,

EXAMPLE 2.2.4: Let X = {a,b,c} & $\tau = \{X, \emptyset, \{a\}, \{a,b\}\}$ be a topology on X. αO(X) = $\{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$. Then {a,c} is α-nhd of c but not nhd of c.

THEOREM 2.2.5:[35] A subset A is α-open iff it is α-nhd of each of its points.

THEOREM 2.2.6: The α-nhd system α-N(x) of a point $x \in X$ satisfies the following properties.

i) α-N(x) $\neq \emptyset \; \forall x \in X$

ii) **[35]** if $N \in$ α-N(x) then $x \in N$

iii) **[35]** $N \in$ α-N(x) & $N \subset M \Rightarrow M \in$ α-N(x)

iv) $N \in$ α-N(x) & $M \in$ α-N(x) $\Rightarrow N \cap M \in$ α-N(x)

v) **[35]** $N \in$ α-N(x) $\Rightarrow \exists M \in$ α-N(x) such that $M \subset N$ and $M \in$ α- N(y) $\forall y \in M$.

THEOREM 2.2.7: For each $x \in X$, let α-N(x) satisfies the following conditions

[P1] $N \in$ α-N(x) $\Rightarrow x \in N$

[P2] $N \in$ α -N(x) $M \in$ α-N(x) $\Rightarrow N \cap M \in$ α-N(x)

Let τ_1 consists of the empty set and all the non-empty subsets A of X having the property that $x \in A \Rightarrow$ there exists an $N \in$ α-N(x) such that $x \in N \subset A$, then τ_1 is a topology for X.

THEOREM 2.2.8: Let A be a subset of X then α-interior of A is the union of all α-nhd subsets of A. i.e., $I_\alpha(A) = \cup\{U \in \alpha\text{-}N(x): U \subset A\}$.

THEOREM 2.2.9: Let A be a subset of X. Let P(X) denote the collection of all possible subsets of a non empty set X, then a closure operator is a mapping C_α of P(X) into itself satisfies the following.

[i] $C_\alpha(\varnothing) = \varnothing$

[ii] $A \subset C_\alpha(A)$

[iii] $A \subset B \Rightarrow C_\alpha(A) \subset C_\alpha(B)$

[iv] [35] $C_\alpha(A \cup B) \subset C_\alpha(A) \cup C_\alpha(B)$

[v] [35] $C_\alpha(C_\alpha(A)) = C_\alpha(A)$

[vi] [35] $C_\alpha(A \cap B) \subset C_\alpha(A) \cap C_\alpha(B)$

THEOREM 2.2.10: Let A and B be subsets of X then the derived sets $D_\alpha(A)$ & $D_\alpha(B)$ have the following properties.

(i) $D_\alpha(\emptyset) = \emptyset$.

(ii) [35] $A \subset B \Rightarrow D_\alpha(A) \subset D_\alpha(B)$

(iii) $x \in D_\alpha(A) \Rightarrow x \in D_\alpha[A - \{x\}]$

(iv) [35] $D_\alpha(A) \cup D_\alpha(B) \subset D_\alpha(A \cup B)$

(v) [35] $D_\alpha(A \cap B) \subset D_\alpha(A) \cap D_\alpha(B)$

THEOREM 2.2.11: $I_\alpha(A)$, $Ext_\alpha(A)$ are disjoint where $A \subset X$ and hence for $x \in A$, $U \in \alpha\text{-}N(x)$ & $Ext_\alpha(A)$ are disjoint

THEOREM 2.2.12: If A is a subset of X and $A \cap Fr_\alpha(A) = \emptyset$ then A is α-nhd of its points.

Converse of theorem 2.2.13 is not true in general, for,

EXAMPLE 2.2.13: Let X = {1,2,3,4,5}

$\tau = \{ Ø, X, \{2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\}\}$ A = $\{1,2,4\}$. Then A is α-nhd of 2. But $A \cap Fr_{\alpha}(A) = \{1,2,4\} \cap \{1,3,4,5\} \neq Ø$.

2.3 SOME MORE PROPERTIES OF αg- NEIGHBOURHOODS OF A POINT & αg-INTERIOR OF A SET

LEMMA 2.3.1: Any arbitrary union of αg-nhds of a point x is again a αg-nhd of x.

NOTE 2.3.2 :Similarly finite intersection of αg-nhd of a point is also a αg-nhd of that point.

THEOREM 2.3.3: Every nhd of x in X is αg-nhd of x.

PROOF: Easy proof

Converse of the theorem 2.3.3 is not true in general, For,

EXAMPLE 2.3.4: Let X = {a,b,c} & $\tau = \{X, Ø, \{a\}, \{a,b\}\}$ be a topology on X. αgO(X) = $\{X, Ø, \{a\}, \{a,b\}, \{a,c\}\}$. Then {a,c} is αg-nhd of c but not nhd of c.

LEMMA 2.3.5: Let $\{B_i \mid i \in I\}$ be a collection of αg-open sets in X. Then $\bigcup_{i \in I} B_i \in \alpha gO(X)$

THEOREM 2.3.6: A subset A is αg-open iff it is αg-nhd of each of its points.

THEOREM 2.3.7: The αg-nhd system αg-N(x) of a point $x \in X$ satisfies the following properties.

i) αg-N(x) $\neq \emptyset$ $\forall x \in X$

ii) if $N \in$ αg-N(x) then $x \in N$

iii) $N \in \alpha$g-N(x) & $N \subset M \Rightarrow M \in \alpha$g-N(x)

iv) $N \in \alpha$g-N(x) & $M \in \alpha$g-N(x) $\Rightarrow$ $N \cap M \in \alpha$g-N(x)

v) $N \in \alpha$g-N(x)$\Rightarrow \exists M \in \alpha$g-N(x) such that $M \subseteq N$ and $M \in \alpha$g-N(y) $\forall$ y $\in M$

THEOREM 2.3.8: For each $x \in X$, let αg-N(x) satisfies the following conditions

[P1] $N \in \alpha$g-N(x) $\Rightarrow x \in N$

[P2] $N \in \alpha$g -N(x) $M \in \alpha$g-N(x) $\Rightarrow N \cap M \in \alpha$g-N(x)

Let τ_1 consists of the empty set and all the non-empty subsets A of X having the property that $x \in A \Rightarrow$ there exists an $N \in \alpha$g-N(x) such that $x \in N \subset A$, then τ_1 is a topology for X.

EXAMPLE 2.3.9: Let X = {a,b,c}, τ= { Ø, {a},{b},{a,b},X}. Then αgO(X)={ϕ,{a},{b},{a,b},{a,c},{b,c},X}. Let α-N(a)={a}, α-N(b) = {{b},{a,b}}, α -N(c) = {{a,c},{b,c}, X} be any collection of subsets of X. We observe that α-N(a), α-N(b), α-N(c) satisfy [P1] & [P2] for, Consider α-N(b), {a,b}$\in$ α-N(b)$\Rightarrow b \in$ {a,b} which is [P1]. Also {a,b}$\cap$ {b}={b}$\in \alpha$-N(b) which is [P2]. Similarly we can show that α-N(a) & α-N(c) satisfies [P1] & [P2]. Now we find the members of τ_1 as follows. (1) By definition Ø$\in \tau_1$. Also $X \in \tau_1$ because $a \in X \Rightarrow$ there exists $X \in \alpha$-N(c) such that that $a \in X \subset X$, $b \in X \Rightarrow \exists X \in \alpha$-N(c), such that $b \in X \subset X$, $c \in X \Rightarrow \exists$ $X \in \alpha$-N(c) such that $c \in X \subset X$. Now (i) (a,c)$\in \tau_1$ because $a \in$ {a,c}$\Rightarrow \exists$ {a}$\in \alpha$-N(a) such that $a \in$ {a}$\subset${a,c} $c \in$ {a,c}$\Rightarrow \exists$ {a,c}$\in \alpha$-N(c) such that $c \in$ {a,c} $\subset$ {a,c}. (ii) {c} $\notin \tau_1$ because $c \in$ {c}$\Rightarrow \exists$ {a,c}$\in \alpha$-N(c) such that $c \in$ {a,c} $\not\subset$ {c}. (iii) {a,b}$\in \tau_1$ because $a \in$ {a, b}, there exist {a} in α-N(a) which contains a and is contained in {a,b}. Similarly $b \in \{a,b\}$, there exist {b} in α-N(b) which contains b and is contained in {a,b}. (iv) $\{b,c\} \notin \tau_1$ because for $b \in \{b,c\}$ $\Rightarrow$ there exist $\{a,b\} \in$α-N(b) such that $\{b\} \in \{a,b\} \not\subset \{b,c\}$. Hence $\tau_1 = \{Ø, \{a\}, \{b\}, \{a,b\},$ {a,c},X} which is a topology.

COROLLARY 2.3.10: If A is a αg-closed subset of X and $x \in X - A$, then there exists a αg-nhd N of x such that $N \cap A = \varnothing$.

NOTE 2.3.11: Since every α-open set is αg-open set, every α-interior point of a set $A \subset X$ is αg-interior point of A. Thus $I_\alpha(A) \subset I_{\alpha g}(A)$. In general $I_{\alpha g}(A) \neq I_\alpha(A)$ which is shown in the following example

EXAMPLE 2.3.12: Let X = {a, b, c,d} and τ={ Ø, {a},{a,b},{a,b,d},X} be a topology. Here αOX = { Ø, {a},{a, b},{a, b, c}, {a, b, d}, X} and αF(X) = { Ø,{c},{d}, {c,d}, {b,c,d}, X}. Let A_1 = {a,c,d}. Then A_1 is αg-closed sets since $C_\alpha(A_1) = X \subset X$. So complement of A_1 ={b} is αg-open in X. Now let A = {b}. Then $I_{\alpha g}(A)$ = {b}. But $I_\alpha(A)$ ={ Ø }. So $I_{\alpha g}(A_1) \neq I_\alpha(A_1)$.

THEOREM 2.3.13: A subset A of X is αg-open iff $A = I_{\alpha g}(A)$.

LEMMA 2.3.14: If A and B are subsets of X and $A \subset B$ then $I_{\alpha g}(A) \subset I_{\alpha g}(B)$.

NOTE 2.3.15: $I_{\alpha g}(A) = I_{\alpha g}(B)$ does not imply that A = B. This is shown in the following example.

EXAMPLE 2.3.16: Let X = {a, b, c} and τ= { Ø, {a, b}, X} be a topology. Let A ={a} and B ={a,c}, then $I_{\alpha g}(A) = I_{\alpha g}(B)$ but $A \neq B$.

THEOREM 2.3.17: Let A and B be subsets of X. Then

(i) $I_{\alpha g}(A) \cup I_{\alpha g}(B) \subset I_{\alpha g}(A \cup B)$

(ii) $I_{\alpha g}(A \cap B) \subset I_{\alpha g}(A) \cap I_{\alpha g}(B)$

THEOREM 2.3.18: Let A be a subset of X then αg-interior of A is the union of all αg-nhd subsets of A. i.e., $I_{\alpha g}(A) = \cup\{U \in \alpha g\text{-}N(x): U \subset A\}$.

2.4. PROPERTIES OF αg- CLOSURE, αg–DERIVATIVE SETS αg-EXTERIOR, αg-FRONTIER, αg- BOUNDARY SET

THEOREM 2.4.1 : Let A be a subset of X. Let P(X) denote the collection of all possible subsets of a non empty set X, then a closure operator is a mapping $C_{\alpha g}$ of P(X) into itself satisfies the following.

[C1] $C_{\alpha g}(\varnothing) = \varnothing$

[C2] $A \subset C_{\alpha g}(A)$

[C3] $A \subset B \Rightarrow C_{\alpha g}(A) \subset C_{\alpha g}(B)$

[C4] $C_{\alpha g}(A \cup B) \subset C_{\alpha g}(A) \cup C_{\alpha g}(B)$

[C5] [$C_{\alpha g}(A \cap B) \subset C_{\alpha g}(A) \cap C_{\alpha g}(B)$

[C6] $C_{\alpha g}(C_{\alpha g}(A)) = C_{\alpha g}(A)$

NOTE 2.4.2: Equality does not hold in Theorem 2.4.1 [C4], which is shown by the following.

EXAMPLE 2.4.3: X= {a,b,c,d} τ = {∅, {a},{a,d},{b,c},{a,b,c},{b,c,d}, X} be a topology of X. Now αgC(X) = {∅, {a}, {b}, {c}, {d}, {a,d}, {b,c},{b,d},{b,c,d},{a,c,d},X}. Take A = {a} and B ={b} then A∪B = {a,b}. Now, we obtain that $C_{\alpha g}(A)$ ={a}, $C_{\alpha g}(B)$ = {b} and $C_{\alpha g}(A \cup B)$ = X. It follows that $C_{\alpha g}(A) \cup C_{\alpha g}(B) \neq C_{\alpha g}(A \cup B)$.

THEOREM 2.4.4: Let $C_{\alpha g}$ be a closure operator defined on X satisfying the properties of Theorem 2.4.1. Let $\mathcal{F}$ be the family of all subsets F of X for which $C_{\alpha g}(F) = F$ and τ be the family of all complements of members of $\mathcal{F}$, then τ is topology for X such that for any arbitrary subset A of X, $A=C_{\alpha g}(A)$.

DEFINITION 2.4.5: Let (X, τ) be a topological space and A be a subset of X. Then a point $x \in X$ is called a **αg-limit point of A** if and only if every αg-nhd. of x contains a point of A distinct from x. i.e., $(N - \{x\}) \cap A \neq \emptyset$ $\forall$ αg-nhd. N of x. Alternatively, we can say that a point $x \in X$ is called a αg-limit point of A if and only if every αg-open set G containing x contains a point of A other than x.

The set of all αg-limit points of A is called a **αg-derived set** of A and is denoted by $D_{\alpha g}(A)$.

EXAMPLE 2.4.6: Let (X, D) be any discrete topological space and let A be any subset of X. Then no point $x \in X$ can be a αg-limit point of A since {x} is a αg-open set which contains no points of A other than (possibly) x. Thus $D_{\alpha g}(A) = \emptyset$.

EXAMPLE 2.4.7: Consider any indiscrete topological space (X,I). Let A be a subset of X containing two or more points of X. Then every point $x \in X$ is a αg-limit point of A since the only αg-open set containing x is X which contains all the points of A and must therefore contains a point of A other than x (Since A contains more than one point). Hence $D_{\alpha g}(A)=X$.

If A = {p] consisting of a single point p. Then all the points of X other than p are αg-limit points of A, for $x \neq p \in X$, then the only αg-open set containing x is X which contains the point p of A which is different from x. Hence $D_{\alpha g}(A) = X-\{p\}$.

If $A = \emptyset$, then evidently no point of X can be a αg-limit point of A and so $D_{\alpha g}(A) = \emptyset$

THEOREM 2.4.8: Let A and B be subsets of X then the derived sets $D_{\alpha g}(A)$. & $D_{\alpha g}(B)$ have the following properties.
(1) $D_{\alpha g}(\emptyset) = \emptyset$. (2) $A \subset B \Rightarrow D_{\alpha g}(A) \subset D_{\alpha g}(B)$ (3) $x \in D_{\alpha g}(A) \Rightarrow x \in D_{\alpha g}[A - \{x\}]$ (4) $D_{\alpha g}(A) \cup D_{\alpha g}(B) \subset D_{\alpha g}(A \cup B)$ (5) $D_{\alpha g}(A \cap B) \subset D_{\alpha g}(A) \cap D_{\alpha g}(B)$.

NOTE 2.4.9: Equality does not hold in Theorem 2.4.8(4), which are shown by the following.

EXAMPLE 2.4.10: X= {a,b,c,d} $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, X\}$ be a topology of X. Then $\alpha gO(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X\}$. Take A = {a,c,d} and B ={d} then $A \cup B = X$. Now, we obtain that $D_{\alpha g}(A) = \{d\}$, $D_{\alpha g}(B) = \{\ \}$ and $D_{\alpha g}(A \cup B) = X$. It follows that $\{d\} = D_{\alpha g}(A) \cup D_{\alpha g}(B) \neq D_{\alpha g}(A \cup B) = X$.

NOTE 2.4.11: $D_{\alpha g}(A) = D_{\alpha g}(B)$ does not imply A = B. For,

EXAMPLE 2.4.12: Consider X= {a,b,c,d} $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, X\}$ be a topology of X. If A = {c,d} and B = {d}. Then $D_{\alpha g}(A) = D_{\alpha g}(B) = \emptyset$. Hence the example.

THEOREM 2.4.13: Let A be a subset of X. Then A is αg-closed if and only it contains the set of its αg-limit points.
PROOF: By definition, X-A is αg-open, if A is αg-closed set. Thus, A is αg-closed if and only if for every point in X-A has a αg-nhd., contained

in X-A i.e., no point of X-A is αg-limit point of A, which is equivalently saying that A contains each of its αg-limit points.

THEOREM 2.4.14: Let (X, τ) be a topological space and A be a subset of X. Then A is αg-closed if and only if $D_{\alpha g}(A) \subset A$.

THEOREM 2.4.15: Let A be any subset of a topological space (X, τ). Then $A \cup D_{\alpha g}(A)$ is αg-closed set.

THEOREM 2.4.16: In any topological space (X, τ), every $D_{\alpha g}(A)$ is a αg-closed set.

THEOREM 2.4.17: $I_{\alpha g}(A)$, $Ext_{\alpha g}(A)$ are disjoint where $A \subset X$ and hence for $x \in A$, $U \in$ αg-N(x) & $Ext_{\alpha g}(A)$ are disjoint.

THEOREM 2.4.18: If A is a subset of X and $A \cap Fr_{\alpha g}(A) = \emptyset$ then A is αg-nhd of its points.

Converse of theorem 2.4.18 is not true in general, for,

EXAMPLE 2.4.19: Let X = {1,2,3,4,5}

τ= { Ø,X,{2},{3,4},{2,3,4},{1,3,4},{1,2,3,4}} A = {1,2,4}. Then A is αg-nhd of 2. But $A \cap Fr_{\alpha g}(A) = \{1,2,4\} \cap \{1,3,4,5\} \neq \emptyset$.

DEFINITION 2.4.20: (i) The set $b_{\alpha g}$ = A- $I_{\alpha g}(A)$ is said to **αg-Boundary of A** (ii) The set $C_{\alpha g}$ - $I_{\alpha g}$ is called the **αg -frontier** of A is denoted by $F_{\alpha g}(A)$.

THEOREM 2.4.21: For a subset A of a space X, the following statements hold:

(1) $b_{\alpha g}(A) \subset b(A)$ where b(A) denotes the boundary of A (2) A = $I_{\alpha g}(A) \cup b_{\alpha g}(A)$ (3) $I_{\alpha g}(A) \cap b_{\alpha g}(A) = \emptyset$ (4) A is an αg–open set if and only if $b_{\alpha g}(A) = \emptyset$ (5) $b_{\alpha g}(I_{\alpha g}(A)) = \emptyset$ (6) $I_{\alpha g}(b_{\alpha g}(A)) = \emptyset$ (7) $b_{\alpha g}(b_{\alpha g}(A)) = b_{\alpha g}(A)$ (8) $b_{\alpha g}(A))$ = A $\cap Cl_{\alpha g}$(X-{A} (9) $b_{\alpha g}(A) = D_{\alpha g}$(X-A)

The converse of the theorem 2.4.21 (1) is not true in general

EXAMPLE 2.4.22: X={a,b,c} τ = { Ø, {a}, {a,b}, X}. If A = {a,c}, then $b_{\alpha g}(A)$ = Ø and b(A) = {c}. Hence $b(A) \not\subset b_{\alpha g}(A)$

THEOREM 2.4.23: For a subset A of a space X, the following statements hold.

1) $Fr_{\alpha g}(A) \subset Fr(A)$ where Fr(A) denotes the frontier of A

2) $C_{\alpha g}(A) = I_{\alpha g}(A) \cup Fr_{\alpha g}(A)$

3) $I_{\alpha g}(A) \cap Fr_{\alpha g}(A) = \varnothing$

4) $b_{\alpha g}(A) \subset Fr_{\alpha g}(A)$

5) $Fr_{\alpha g}(A) = b_{\alpha g}(A) \cup D_{\alpha g}(A)$

6) A is an αg–open set if and only if $Fr_{\alpha g}(A) = D_{\alpha g}(A)$

7) $Fr_{\alpha g}(A) = C_{\alpha g}(A) \cap C_{\alpha g}(X\text{-}A)$

8) $Fr_{\alpha g}(A) = Fr_{\alpha g}(X\text{-}A)$

9) $Fr_{\alpha g}(A)$ is αg-closed

10) $Fr_{\alpha g}(Fr_{\alpha g}(A)) \subset Fr_{\alpha g}(A)$;

11) $Fr_{\alpha g}(I_{\alpha}(A)) \subset Fr_{\alpha g}(A)$;

12) $Fr_{\alpha g}(C_{\alpha g}(A)) \subset Fr_{\alpha g}(A)$;

13) $I_{\alpha g}(A) = A - Fr_{\alpha g}(A)$;

The converse of (1) and (4) of Theorem 2.4.23 are not true in general, as shown by the following example

EXAMPLE 2.4.24: X = {a,b,c} τ = { Ø, {a}, {a,b}, X}. Then αgO(X) = {Ø,{a},{a,b},{a,c},X} & αgC(X) ={Ø,{b},{c},{b,c},{a,c},X}. If A = {b}, B = {a, b}, then Fr(A) = {b, c} $\not\subset$ {b} = $Fr_{\alpha g}$ (A), $Fr_{\alpha g}(B)$ ={c} $\not\subset$ $b_{\alpha g}$ (B) =$\varnothing$.

2.5 $\gamma_{g\alpha g}$ - OPEN SETS

DEFINITION 2.5.1: A subset A of a topological space X is called $\gamma_{g\alpha g}$ - open set iff $A \cap B \in GPO(X)$ for every $B \in \alpha gO(X)$.

The family of all $\gamma_{g\alpha g}$-open subset of X is denoted by $\gamma_{g\alpha g}O(X)$. The complement of a $\gamma_{g\alpha g}$-open set is called $\gamma_{g\alpha g}$-closed set. The family of $\gamma_{g\alpha g}$- closed subsets of X is denoted by $\gamma_{g\alpha g}C(X)$.

THEOREM 2.5.2: $\alpha gO(X) \subset \gamma_{g\alpha g}O(X)$

PROOF: We know that $\alpha gO(X) \subset GPO(X)$. Now for all $A \in \alpha gO(X)$, $A \cap X = A \in \alpha gO(X) \subset GPO(X) \Rightarrow A \cap X = A \in GPO(X) \Rightarrow A \in \gamma_{g\alpha g}O(X)$.

REMARK 2.5.3: The following two example illustrates that $\alpha gO(X) \subset \gamma_{g\alpha g}O(X) \subset GPO(X)$.

EXAMPLE 2.5.4: (i) Let X = {a,b,c,d}, τ ={Ø,{a,b},{a,b,c},X}

$\alpha O(X)$ = {Ø,{a,b},{a,b,c},{a,b,d}, X}.

$\alpha C(X)$ = {Ø,{c,d},{d},{c},X}.

$\alpha gC(X)$ ={Ø,{b,c,d},{a,c,d},{a,b,d},{c,d},{a,d},{b,d},{d},{c},X}.

$\alpha gO(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{b,c},{a,b,c},{a,b,d},X}.

PO(X) ={Ø,{a},{b} {a,b},{a,c},{a,d},{b,c},{b,d},{a,b,c},{a,b,d},{a,c,d}, {b,c,d},X}.

PC(X) ={Ø,{a},{b},{c},{d},{a,c},{a,d},{c,d},{b,c},{b,d},{a,c,d}, {b,c,d},X}.

GPC(X) ={Ø,{a},{b},{c},{d},{a,c},{a,d},{c,d},{b,c},{b,d},{a,b,d}, {a,c,d},{b,c,d},X}.

GPO(X) ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{b,c},{b,d},{a,b,c}, {a,b,d},{a,c,d},{b,c,d},X}.

$\gamma_{g\alpha g}O(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{b,c},{b,d},{a,b,c},
{a,b,d},{a,c,d},{b,c,d},X}.

So $\alpha gO(X) \subset \gamma_{g\alpha g}O(X)$.

(ii) Let X = {a,b,c,d}, τ ={Ø,{a,d},{a,b,c},X}

αO(X) = {Ø,{a,d},{a,b,c},{a,b,d},{a,c,d},X}.

αC(X) = {Ø,{b,c},{d},{c},{b},X}.

αgC(X) ={Ø,{b,c,d},{a,c,d},{a,b,d},{c,d},{b,c},{b,d},{b},{c},{d},X}.

αgO(X)={Ø,{a},{b},{c},{a,b},{a,c},{b,d},{a,b,c},{a,b,d},{a,c,d},X}.

PO(X) ={Ø,{a},{a,b},{a,c},{a,d},{c,d},{b,d},{a,b,c},{a,b,d},{a,c,d},
{b,c,d},X}.

PC(X) ={Ø,{a},{b},{c},{d},{a,b},{a,c},{c,d},{b,c},{b,d},{b,c,d},X}.

GPC(X) ={Ø,{a},{b},{c},{d},{a,b},{a,c},{c,d},{b,c},{b,d},{a,b,d},
{a,c,d},{b,c,d},X}.

GPO(X) ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{c,d},{b,d},{a,b,c},{a,b,d},
{a,c,d},{b,c,d},X}.

$\gamma_{g\alpha g}O(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{a,b,c},{a,b,d},{a,c,d},
{b,c,d},X}.

So $\gamma_{g\alpha g}O(X) \subset GPO(X)$.

THEOREM 2.5.5: $\gamma_gO(X) \subset \gamma_{g\alpha g}O(X)$

PROOF: We have $\gamma_gO(X) \subset GPO(X)$. Let $A \in \gamma_gO(X)$. Then $A \in GPO(X)$. Also for all $B \in \alpha gO(X) \subset GPO(X)$, $A \cap B \in GPO(X) \Rightarrow A \in \gamma_{g\alpha g}O(X)$. Hence the result.

REMARK 2.5.6: The following example illustrates that both $\alpha gO(X)$ & $\gamma_gO(X)$ are the subsets of $\gamma_{g\alpha g}O(X)$.

EXAMPLE 2.5.7: Let X= {a,b,c,d}, τ ={Ø,{a},{a,b},{a,b,c},X}

αgO(X) ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{b,c},{a,b,c},{a,b,d},{a,c,d}, X}.

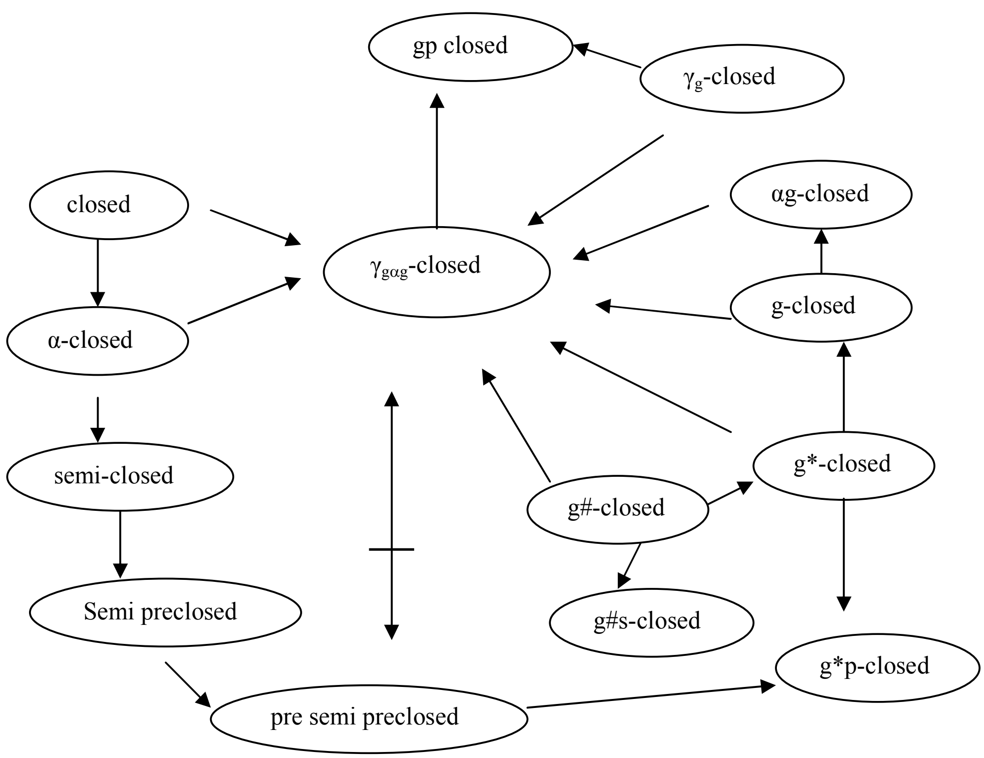
GPO(X) ={Ø, {a}, {b}, {c}, {a,b},{a,c}, {a,d}, {b,c},{b,d}, {a,b,c}, {a,b,d}, {a,c,d}, X}.

γ_gO(X) ={Ø,{a},{b},{c},{a,b},{a,c},{b,c},{a,b,c},{a,b,d},{a,c,d}, X}.

$\gamma_{g\alpha g}$O(X) ={Ø,{a},{b},{c},{a,b},{a,c},{a,d}, {b,c}, {a,b,c}, {a,b,d}, {a,c,d}, X}.

$\therefore$ $\alpha gO(X) \subset \gamma_{g\alpha g}O(X)$ & $\gamma_gO(X) \subset \gamma_{g\alpha g}O(X)$.

REMARK 2.5.8: The following diagram establishes the relationships between $\gamma_{g\alpha g}$ closed sets and some other sets. A $\longrightarrow$ B(resp A $\not\longrightarrow$ B. A $\not\longleftrightarrow$ B) represents A implies B but not conversely (resp. A need not imply B, A and B are independent of each other)



THEOREM 2.5.9: For a topological space X, every singleton subset of X is αg-open if it is $\gamma_{g\alpha g}$-open.

PROOF: Suppose every singleton subset of X is αg-open. Then for $x \in X$ and any subset B of X, $\{x\} \cap B$ is either $\{x\}$ or Ø, both which are gp-open. Hence $\{x\}$ is in $\gamma_{g\alpha g}O(X)$.

REMARK 2.5.10: For a topological space X if a singleton subset of X is $\gamma_{g\alpha g}$-closed then it is not necessary that it is αg-closed. This is illustrated in the following example.

EXAMPLE 2.5.11: Let X = {a,b,c,d}, τ ={Ø,{a},{a,d},{a,b,c},X}

αgC(X) ={Ø,{b,c,d},{a,c,d},{a,b,d},{c,d},{b,d},{b,c},{d},{c},{b}, X}.

$\gamma_{g\alpha g}O(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{a,b,c},{a,b,d},{a,c,d}, X}.

Here {a} is $\gamma_{g\alpha g}$-closed set but it is not αg-closed set.

2.6 PROPERTIES OF $\gamma_{g\alpha g}$-NEIGHBOURHOODS

DEFINITION 2.6.1: Let X be a topological space and $A \subset X$. Then A is called a **$\gamma_{g\alpha g}$- neighbourhood**, denoted by $\gamma_{g\alpha g}$-nhd, of a point x in X, if there exist a $\gamma_{g\alpha g}$-open set U in X such that $x \in U \subset A$. The $\gamma_{g\alpha g}$-nhd system of a point $x \in X$ is denoted by $\gamma_{g\alpha g}$-N(x).

NOTE 2.6.2: Clearly every $\gamma_{g\alpha g}$-open set A is $\gamma_{g\alpha g}$-nhd of the points of A.

LEMMA 2.6.3: Any arbitrary union of $\gamma_{g\alpha g}$-nhds of a point x is again a $\gamma_{g\alpha g}$-nhd of x.

PROOF: Let $\{A_\lambda\}_{\lambda \in I}$ be the arbitrary collection of $\gamma_{g\alpha g}$-nhds of a point $x \in X$. We have to prove that $\cup A_\lambda$ for $\lambda \in I$ also a $\gamma_{g\alpha g}$-nhd of x. Now there exist $\gamma_{g\alpha g}$-open set M_x such that $x \in M_x \subset A_\lambda \subset \cup A_\lambda$ for $\lambda \in I$. i.e., $x \in M_x \subset \cup A_\lambda$ for $\lambda \in I$. Therefore $\cup A_\lambda$ for $\lambda \in I$ is a $\gamma_{g\alpha g}$-nhd of x. i.e., arbitrary union of $\gamma_{g\alpha g}$-nhd of a point x is again a $\gamma_{g\alpha g}$-nhd of x.

NOTE 2.6.4: Similarly finite intersection of $\gamma_{g\alpha g}$-nhd of a point is also a $\gamma_{g\alpha g}$-nhd of that point.

THEOREM 2.6.5: Every nhd of x in X is $\gamma_{g\alpha g}$-nhd of x.

Converse of the theorem 2.6.5 is not true in general, For,

EXAMPLE 2.6.6: Let X = {a,b,c} & $\tau = \{X, \emptyset, \{a\}, \{a,c\}\}$ be a topology on X. Then $\gamma_{g\alpha g}O(X) = \{X, \emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}\}$ Then {a,b} is $\gamma_{g\alpha g}$ -nhd of b but not nhd of b.

LEMMA 2.6.7: Let $\{B_i \mid i \in I\}$ be a collection of $\gamma_{g\alpha g}$-open sets in X. Then $\bigcup_{i \in I} B_i \in \gamma_{g\alpha g}O(X)$.

PROOF: Easy to prove

THEOREM 2.6.8: A subset A is $\gamma_{g\alpha g}$-open iff it is $\gamma_{g\alpha g}$-nhd of each of its points.

PROOF: Let $A \subset X$ be a $\gamma_{g\alpha g}$-open set. Then for each $x \in A$, $x \in A \subset A$ and A is $\gamma_{g\alpha g}$-open $\Rightarrow$ A is $\gamma_{g\alpha g}$-nhd of x. Conversely suppose U is $\gamma_{g\alpha g}$-nhd of each of its points. Then for each $x \in U$, there exists $N_x \in \gamma_{g\alpha g}O(X)$ such that $N_x \subset U$. Then $U = \cup \{N_x \mid x \in X\}$. Since each N_x is $\gamma_{g\alpha g}$-open it follows that U is $\gamma_{g\alpha g}$-open by lemma 2.6.7.

THEOREM 2.6.9: The $\gamma_{g\alpha g}$-nhd system $\gamma_{g\alpha g}$-N(x) of a point $x \in X$ satisfies the following properties.

i) $\gamma_{g\alpha g}$-N(x) $\neq \emptyset \quad \forall x \in X$

ii) if $N \in \gamma_{g\alpha g}$-N(x) then $x \in N$

iii) $N \in \gamma_{g\alpha g}$-N(x) & $N \subset M \Rightarrow M \in \gamma_{g\alpha g}$-N(x)

iv) $N \in \gamma_{g\alpha g}$-N(x), $M \in \gamma_{g\alpha g}$-N(x) $\Rightarrow N \cap M \in \gamma_{g\alpha g}$-N(x)

v) $N \in \gamma_{g\alpha g}$-N(x)$\Rightarrow \exists\ M \in \gamma_{g\alpha g}$-N(x) such that $M \subset N$ and $M \in \gamma_{g\alpha g}$-N(y) $\forall y \in M$

PROOF: (i) is trivial

(ii) Let $N \in \gamma_{g\alpha g}$-N(x)$\Rightarrow$N is the $\gamma_{g\alpha g}$-nhd of x $\Rightarrow x \in N$.

(iii) Let $N \in \gamma_{g\alpha g}$-N(x) and $N \subset M$. Therefore there exists $U \in \gamma_{g\alpha g}O(X)$ such that $x \in U \subset N \subset M \Rightarrow$ M is $\gamma_{g\alpha g}$-nhd of x$\Rightarrow M \in \gamma_{g\alpha g}$-N(x)

(iv) $N \in \gamma_{g\alpha g}$-N(x) & $M \in \gamma_{g\alpha g}$-N(x) $\Rightarrow \exists\ \gamma_{g\alpha g}$-open sets N_x & M_x such that $x \in N_x \subset N$ & $x \in M_x \subset M$. Now $x \in N_x \cap M_x \subset N \cap M$. By note 2.6.4, the result follows

(v) Let $N \in \gamma_{g\alpha g}$-N(x). Therefore there exists $M \in \gamma_{g\alpha g}O(X)$ such that $x \in M \subset N$. Since every $\gamma_{g\alpha g}$-open is $\gamma_{g\alpha g}$-nhd, M is $\gamma_{g\alpha g}$-nhd of its points $\Rightarrow \forall y \in M$, M is $\gamma_{g\alpha g}$-nhd of y $\Rightarrow M \in \gamma_{g\alpha g}$-N(y) $\forall y \in M$.

COROLLARY 2.6.10: If A is a $\gamma_{g\alpha g}$-closed subset of X and $x \in X - A$, then there exists a $\gamma_{g\alpha g}$-nhd N of x such that $N \cap A = \emptyset$. **PROOF:** If A is a $\gamma_{g\alpha g}$-closed set in X, then X-A is $\gamma_{g\alpha g}$-open set. By theorem 2.6.8, X-A contains a $\gamma_{g\alpha g}$-nhd. of each of its points. This implies that there exists a $\gamma_{g\alpha g}$-nhd. N of x such that $N \subset X-A$. It is clear that no point of N belongs to A and hence $N \cap A = \emptyset$.

THEOREM 2.6.11: For each $x \in X$, let $\gamma_{g\alpha g}$-N(x) satisfies the following conditions

[P1] $N \in \gamma_{g\alpha g}\text{-}N(x) \Rightarrow x \in N$

[P2] $N \in \gamma_{g\alpha g}\text{-}N(x), M \in \gamma_{g\alpha g}\text{-}N(x) \Rightarrow N \cap M \in \gamma_{g\alpha g}\text{-}N(x)$.

Let τ_1 consists of the empty set and all the non-empty subsets of A of X having the property that $x \in A \Rightarrow$ there exists an $N \in \gamma_{g\alpha g}$-N(x) such that $x \in N \subset A$, then τ_1 is a topology for X.

PROOF: 1) By definition $\emptyset \in \tau_1$. Let $x \in X$, then $\gamma_{g\alpha g}$-N(x) being nonempty, there exists an $N \in \gamma_{g\alpha g}$-N(x) so that by [P1] we have $x \in N$. Also N is a subset of X so that $x \in N \subset X$ and therefore $X \in \tau_1$.

2) Let $A \in \tau_1$ and $B \in \tau_1$. We will show that $A \cap B \in \tau_1$. Now choose x to be an arbitrary element of $A \cap B \Rightarrow x \in A$ and $x \in B$. But as $A \in \tau_1$ and $B \in \tau_1$, there exists $N \in \gamma_{g\alpha g}$-N(x) and $M \in \gamma_{g\alpha g}$-N(x) such that $x \in N \subset A$ and $x \in M \subset B$ or $x \in N \cap M \subset A \cap B$. But $N \cap M \in \gamma_{g\alpha g}$-N(x) by [P2], so that $A \cap B \in \tau_1$.

3) Let $\{A_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of τ_1 open sets and we will show that $\cup\{A_\lambda : \lambda \in \Lambda\} \in \tau_1$. Let x be an arbitrary element of $\cup\{A_\lambda : \alpha \in \Lambda\}$ so that $x \in A_\lambda$ for at least one $\lambda \in \Lambda$. Now $x \in A_\lambda$ and $A_\lambda \in \tau_1$, therefore there exist some $N \in \gamma_{g\alpha g}$-N(x) such that $x \in N \subset A_\lambda$. Or $x \in N \subset \cup\{A_\lambda : \lambda \in \Lambda\} \Rightarrow \cup\{A_\lambda : \lambda \in \Lambda\} \in \tau_1$. Thus the collection τ_1 is a topology for X.

EXAMPLE 2.6.12: Let X ={a,b,c} & $\tau = \{X, Ø, \{a\}, \{a,c\}\}$ be a topology on X. Then $\gamma_{g\alpha g}O(X)$={X,Ø,{a},{c},{a,b},{a,c}}. Let $\gamma_{g\alpha g}$-N(a) ={{a},X} $\gamma_{g\alpha g}$-N(b) ={{a,b}}, $\gamma_{g\alpha g}$-N(c) ={{c},{a,c}} be any collection of subsets of X. We observe that $\gamma_{g\alpha g}$-N(a), $\gamma_{g\alpha g}$-N(b), $\gamma_{g\alpha g}$-N(c) satisfy [P1] & [P2] Now we find the members of τ_1 as follows: (i) By definition $Ø \in \tau_1$. Also $X \in \tau_1$ because $a \in X \Rightarrow$ there exists X∈ $\gamma_{g\alpha g}$-N(a) such that that $a \in X \subset X$, $b \in X \Rightarrow \exists X \in$ $\gamma_{g\alpha g}$-N(a), such that $b \in X \subset X$, $c \in X \Rightarrow \exists X \in$ $\gamma_{g\alpha g}$-N(a) such that $c \in X \subset X$. (ii)$\{c\} \in \tau_1$ because $c \in \{c\}$, but there exist {c} in $\gamma_{g\alpha g}$-N(c) containing c and contained in {c}.(iii) Similarly $\{a\} \in \tau_1$. (iv) $\{b\} \notin \tau_1$ since for $b \in \{b\}$ there exist {a,b} in $\gamma_{g\alpha g}$-N(b) containing b but not contained in {b}. (v) (b,c) $\notin \tau_1$ because $b \in \{b,c\}$, but there does not exist any set in $\gamma_{g\alpha g}$-N(b) containing b and contained in {b,c} (vi) $\{a,c\} \in \tau_1$ because $a \in \{a,c\}$ and there exist {a} in $\gamma_{g\alpha g}$-N(a) which contains a and is contained in {a,c}. (vi) Similarly $\{a,b\} \in \tau_1$. Hence τ_1={ Ø ,{a},{c},{a,b},{a,c},X} which is a topology.

DEFINITION 2.6.13: The intersection of all $\gamma_{g\alpha g}$-closed sets containing a set A is called the $\gamma_{g\alpha g}$-closure of A and is denoted by $\gamma_{g\alpha g}Cl(A)$. Thus $\gamma_{g\alpha g}Cl(A) = \cap\{F|A \subset F \text{ and } F \in \gamma_{g\alpha g}Cl(X)\}$.

REMARK 2.6.14: For a subset A of X $\gamma_{g\alpha g}Cl(A)$ is not necessarily $\gamma_{g\alpha g}$-closed set as shown in the following example.

EXAMPLE 2.6.15: Let X = {a,b,c,d}, τ ={Ø,{a},{a,d},{a,b,c},X}
$\gamma_{g\alpha g}O(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{a,b,c},{a,b,d},{a,c,d}, X}.
$\gamma_{g\alpha g}C(X)$ ={Ø,{b},{c},{d},{c,d},{b,d},{b,c},{b,c,d},{a,b,d},{a,c,d}, X}.
Now $\gamma_{g\alpha g}Cl(\{a\})$ = {a,b,d} ∩ {a,c,d} ∩ X = {a,d} which is not $\gamma_{g\alpha g}$-closed.

THEOREM 2.6.16: Let A be a subset of X. Let P(X) denote the collection of all possible subsets of a non empty set X, then a closure operator is a mapping $\gamma_{g\alpha g}Cl$ of P(X) into itself satisfies the following.

[C1] $\gamma_{g\alpha g}Cl\ (\emptyset) = \emptyset$

[C2] $A \subset \gamma_{g\alpha g}Cl\ (A)$

[C3] $A \subset B \Rightarrow \gamma_{g\alpha g}Cl\ (A) \subset \gamma_{g\alpha g}Cl\ (B)$

[C4] $\gamma_{g\alpha g}Cl\ (A \cup B) \subset \gamma_{g\alpha g}Cl\ (A) \cup \gamma_{g\alpha g}Cl\ (B)$

[C5] $\gamma_{g\alpha g}Cl\ (\gamma_{g\alpha g}Cl\ (A)) = \gamma_{g\alpha g}Cl\ (A)$.

PROOF: Easy to prove.

THEOREM 2.6.17: Let $\gamma_{g\alpha g}Cl$ be a closure operator defined on X satisfying the properties of Theorem 2.6.16. Let F be the family of all subsets F of X for which $\gamma_{g\alpha g}Cl$ = F and τ be the family of all complements of members of F, then τ is topology for X such that for any arbitrary subset A of X, $A = \gamma_{g\alpha g}Cl(A)$

PROOF: Proof is similar to the corresponding result under therorem 2.4.4.

REMARK 2.6.18: $\gamma_{g\alpha g}Cl(A)= \gamma_{g\alpha g}Cl(B)$ does not necessarily imply that A = B. This is shown in the following example.

EXAMPLE 2.6.19: Let X = {a,b,c,d}, τ ={Ø,{a},{a,d},{a,b,c},X}
$\gamma_{g\alpha g}C(X)$ ={Ø,{b},{c},{d},{c,d},{b,d},{b,c},{b,c,d},{a,b,d},{a,c,d}, X}.
Now $\gamma_{g\alpha g}Cl(\{a,b\})$ = X and $\gamma_{g\alpha g}Cl(\{a,b,c\})$ =X. But {a,b} ≠ {a,b,c}.

DEFINITION 2.6.20: Let (X, τ) be a topological space and A be a subset of X. Then a point $x \in X$ is called a **$\gamma_{g\alpha g}$-limit point of A** if and only if every $\gamma_{g\alpha g}$-nhd. of x contains a point of A distinct from x. i.e., $(N - \{x\}) \cap A \neq \emptyset \ \forall \gamma_{g\alpha g}$-nhd N of x. Alternatively, we can say that if and only if every $\gamma_{g\alpha g}$-open set G containing x contains a point of A other than x.

The set of all $\gamma_{g\alpha g}$-limit points of A is called a **$\gamma_{g\alpha g}$-derived set** of A and is denoted by $\gamma_{g\alpha g}D(A)$.

EXAMPLE 2.6.21: Let (X, D) be any discrete topological space and let A be any subset of X. Then no point $x \in X$ can be a $\gamma_{g\alpha g}$-limit point of A since {x} is a $\gamma_{g\alpha g}$-open set which contains no points of A other than (possibly) x. Thus $\gamma_{g\alpha g}D(A) = \emptyset$.

EXAMPLE 2.6.22: Consider any indiscrete topological space (X,I). Let A be a subset of X containing two or more points of X. Then every point $x \in X$ is a $\gamma_{g\alpha g}$-limit point of A since the only $\gamma_{g\alpha g}$-open set containing x is X which contains all the points of A and must therefore contains a point of A other than x. (Since A contains more than one point). Hence $\gamma_{g\alpha g}D(A) = X$.

If A = {p} consisting of a single point p. Then all the points of X other than p are $\gamma_{g\alpha g}$-limit points of A, for $x \neq p \in X$, then the only $\gamma_{g\alpha g}$-open set containing x is X which contains the point p of A which is different from x. Hence $\gamma_{g\alpha g}D(A) = X-\{p\}$.

If A = Ø., then evidently no point of X can be a $\gamma_{g\alpha g}$limit point of A and so $\gamma_{g\alpha g}D(A) = \emptyset$.

THEOREM 2.6.23: Let (X, τ) be a topological space and A be a subset of X. Then A is $\gamma_{g\alpha g}$-closed if and only if $\gamma_{g\alpha g}D(A) \subset A$.

PROOF: ($\Rightarrow$) Suppose A is $\gamma_{g\alpha g}$-closed. i.e., X-A is $\gamma_{g\alpha g}$-open. Now we show that $\gamma_{g\alpha g}D(A) \subset A$. Let $x \in \gamma_{g\alpha g}D(A)$ implies x is a $\gamma_{g\alpha g}$-limit point of A. i.e., every $\gamma_{g\alpha g}$-nhd. of x contains a point of A different from x. Now suppose $x \notin A$ so that $x \in$ X-A, which is $\gamma_{g\alpha g}$-open and by definition of $\gamma_{g\alpha g}$-open sets there exists a $\gamma_{g\alpha g}$-nhd. N of x such that $N \subset X-A$. From this we conclude that N contains no point of A, which is a contradiction. Therefore $x \in A$ and hence $\gamma_{g\alpha g}D(A) \subset A$.

($\Leftarrow$) Assume $\gamma_{g\alpha g}D(A) \subset A$. We show that A is a $\gamma_{g\alpha g}$-closed set in (X, τ). Or we show that there exists $\gamma_{g\alpha g}$-nhd.N of x for each $x \in X-A$. Let x be an arbitrary point of X-A so that $x \notin A$. Since $\gamma_{g\alpha g}D(A) \subset A$, $x \notin A$ implies $x \notin \gamma_{g\alpha g}D(A)$. i.e., there exists a $\gamma_{g\alpha g}$-nhd.N of x, which does not contain any point of A. i.e., there exists a $\gamma_{g\alpha g}$-nhd.N of x which consists of only points of X-A. This means that X-A is $\gamma_{g\alpha g}$-open and hence A is $\gamma_{g\alpha g}$-closed.

THEOREM 2.6.24: Let A is any subset of a topological space (X, τ). Then $A \cup \gamma_{g\alpha g}D(A)$ is αg-closed set.

PROOF :($\Rightarrow$)$A \cup \gamma_{g\alpha g}D(A)$ is $\gamma_{g\alpha g}$-closed set in X if $X - [A \cup \gamma_{g\alpha g}D(A)]$ is a $\gamma_{g\alpha g}$-open set in X. But $X - [A \cup \gamma_{g\alpha g}D(A)] = [X - A] \cap [X-\gamma_{g\alpha g}D(A)]$. Thus we prove that $[X - A] \cap [X-\gamma_{g\alpha g}D(A)]$ is $\gamma_{g\alpha g}$-open set in X i.e., it contains a $\gamma_{g\alpha g}$-nhd. of each of its points. Let $x \in [X - A] \cap [X-\gamma_{g\alpha g}D(A)]$ $\Rightarrow$ $x \in [X - A]$ and $x \in [X-\gamma_{g\alpha g}D(A)]$ or $x \notin A$ and $x \notin \gamma_{g\alpha g}D(A)$. Now $x \notin \gamma_{g\alpha g}D(A) \Rightarrow x$ is not a $\gamma_{g\alpha g}$-limit point of A, it follows that there exist $\gamma_{g\alpha g}$-nhd N of x which contains no points of A other than possibly x. But $x \notin A$ so that N contains no point of A and so $N \subset X- A$. Again N is

$\gamma_{g\alpha g}$-open and is a $\gamma_{g\alpha g}$-nhd of each of its points. Therefore no point of N can be a limit point of A. i.e., no point of N can belong to $\gamma_{g\alpha g}D(A)$. Thus $N \subset X - \gamma_{g\alpha g}D(A)$. Thus we have proved that for all $x \in [X - A] \cap [X-\gamma_{g\alpha g}D(A)]$ that there exist $\gamma_{g\alpha g}$-open neighbourhood N of x such that $N \subset [X- A]\cap[X-\gamma_{g\alpha g}D(A)]$ which implies $[X- A]\cap[X-\gamma_{g\alpha g}D(A)]$ is $\gamma_{g\alpha g}$-open set in X. Hence the proof.

THEOREM 2.6.25 : In any topological space (X, τ), every $\gamma_{g\alpha g}D(A)$ is a $\gamma_{g\alpha g}$-closed set.

PROOF : Let A be a set of X and $\gamma_{g\alpha g}D(A)$ is $\gamma_{g\alpha g}$-derived set of A. Then by theorem 2.6.23, A is $\gamma_{g\alpha g}$-closed set if and only if $\gamma_{g\alpha g}D(A) \subset A$. Hence $\gamma_{g\alpha g}D(A)$ is $\gamma_{g\alpha g}$-closed if and only if $\gamma_{g\alpha g}D(\gamma_{g\alpha g}D(A) \subset \gamma_{g\alpha g}D(A$. i.e., every $\gamma_{g\alpha g}$-limit point of $\gamma_{g\alpha g}D(A)$ belongs to $\gamma_{g\alpha g}D(A)$.

Let x be a $\gamma_{g\alpha g}$-limit point of $\gamma_{g\alpha g}D(A)$. i.e., $x\in \gamma_{g\alpha g}D(\gamma_{g\alpha g}D(A))$ so that there exists a $\gamma_{g\alpha g}$-open set U containing x such that $(U - \{x\})\cap(\gamma_{g\alpha g}D(A)) \neq \emptyset \Rightarrow (U -\{x\}) \cap A \neq \emptyset$ since every $\gamma_{g\alpha g}$-nhd of an element of $\gamma_{g\alpha g}D(A)$ has at least one point of A. Therefore, x is a $\gamma_{g\alpha g}$-limit point of A. i.e., $x\in\gamma_{g\alpha g}D(A)$. Thus $x\in\gamma_{g\alpha g}D(\gamma_{g\alpha g}D(A) \Rightarrow x\in \gamma_{g\alpha g}D(A)$. Therefore $\gamma_{g\alpha g}D(A)$ is $\gamma_{g\alpha g}$-closed set in X.

DEFINITION 2.6.26: The union of all $\gamma_{g\alpha g}$-open sets which are contained in A is called the $\gamma_{g\alpha g}$-interior of A and is denoted by $\gamma_{g\alpha g}Int(A)$. i.e., $\gamma_{g\alpha g}Int(A) = \cup \{ U|U\in A \text{ and } U\in \gamma_{g\alpha g}O(X)\}$

DEFINITION 2.6.27: The union of all $\gamma_{g\alpha g}$-open sets contained in the complement of A is called the $\gamma_{g\alpha g}$-exterior of A, denoted by $\gamma_{g\alpha g}Ext(A)$.

DEFINITION 2.6.28 : The set $\gamma_{g\alpha g}Cl(A) - \gamma_{g\alpha g}Int(A)$ is called the $\gamma_{g\alpha g}$ - frontier of A is denoted by $\gamma_{g\alpha g}Fr(A)$.

DEFINITION 2.6.29: The set $\gamma_{g\alpha g}b(A) = A - \gamma_{g\alpha g}Int(A)$ is said to **$\gamma_{g\alpha g}$-Boundary of A.**

THEOREM 2.6.30: Let A be a subset of X then $\gamma_{g\alpha g}$-interior of A is the union of all $\gamma_{g\alpha g}$-nhd subsets of A. i.e., $\gamma_{g\alpha g}Int(A) = \cup\{U \in \gamma_{g\alpha g}\text{-}N(x): U \subset A\}$.

PROOF : Let $x \in \gamma_{g\alpha g}Int(A)$ then $\exists$ $\gamma_{g\alpha g}$-open set U of A such that $x \in U \subset A$. Since every $\gamma_{g\alpha g}$-open set is a $\gamma_{g\alpha g}$-nhd, $U \in \gamma_{g\alpha g}\text{-}N(x)$ such that $x \in U \subset A$. and so $x \in \cup\{U \in \gamma_{g\alpha g}\text{-}N(x) : U \subset A\}$. Hence $\gamma_{g\alpha g}Int(A) \subset \cup\{U \in \gamma_{g\alpha g}\text{-}N(x) : U \subset A\}$. Now let $x \in \cup\{U \in \gamma_{g\alpha g}\text{-}N(x) : U \subset A\} \Rightarrow x \in U \in \gamma_{g\alpha g}\text{-}N(x)$, which is contained in A for some $U \Rightarrow$ there exist a $\gamma_{g\alpha g}$-open set V in X such that $x \in V \subset U \subset A \Rightarrow x \in \gamma_{g\alpha g}Int(A)$. $\Rightarrow \cup\{U \in \gamma_{g\alpha g}\text{-}N(x) : U \subset A\} \subset I_{\alpha}(A)$. Hence the result.

NOTE 2.6.31: Since every γ_g-open set is $\gamma_{g\alpha g}$-open set, every γ_g-interior point of a set $A \subset X$ is $\gamma_{g\alpha g}$-interior point of A. Thus $\gamma_g Int(A) \subset \gamma_{g\alpha g}Int(A)$. In general $\gamma_{g\alpha g}Int(A) \neq \gamma_g Int(A)$ which is shown in the following example

EXAMPLE 2.6.32: Let X = {a,b,c,d}, τ ={Ø,{a},{a,b},{a,b,c},X}

$\gamma_g O(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{b,c},{a,b,c},{a,b,d},{a,c,d}, X}.

$\gamma_{g\alpha g}O(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{a,d}, {b,c}, {a,b,c}, {a,b,d}, {a,c,d}, X}.

Let A = {a,d}. Then $\gamma_g Int(A)$ ={a} but $\gamma_{g\alpha g}Int(A)$ ={a,d}

$\therefore \gamma_g Int(A) \neq \gamma_{g\alpha g}Int(A)$.

THEOREM 2.6.33: A subset A of X is $\gamma_{g\alpha g}$-open iff $A = \gamma_{g\alpha g}Int(A)$.

PROOF: ($\Rightarrow$) Let A be an $\gamma_{g\alpha g}$-open set. Now A is the largest $\gamma_{g\alpha g}$-open set contained in A. But $\gamma_{g\alpha g}\text{Int}(A)$ is the largest $\gamma_{g\alpha g}$-open set of A. Hence $A = \gamma_{g\alpha g}\text{Int}(A)$.

($\Leftarrow$) Let $A = \gamma_{g\alpha g}\text{Int}(A)$. Then by definition $\gamma_{g\alpha g}\text{Int}(A)$ is an $\gamma_{g\alpha g}$-open set. This implies that A is also $\gamma_{g\alpha g}$-open set. Hence the result.

LEMMA 2.6.34: If A and B are subsets of X and $A \subseteq B$ then $\gamma_{g\alpha g}\text{Int}(A) \subset \gamma_{g\alpha g}\text{Int}(B)$.

PROOF: Easy proof.

NOTE 2.6.35: $\gamma_{g\alpha g}\text{Int}(A) = \gamma_{g\alpha g}\text{Int}(B)$ does not imply that A = B. This is shown in the following example.

EXAMPLE 2.6.36: Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a\},\{a,d\},\{a,b,c\},X\}$ $\gamma_{g\alpha g}O(X) = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}, X\}$. Now $\gamma_{g\alpha g}\text{Int}(\{b,c\}) = \{b\}$ and $\gamma_{g\alpha g}\text{Int}(\{b,c,d\}) = \{b\}$. But $\{b,c\} \neq \{b,c,d\}$.

THEOREM 2.6.37: Let A and B be subsets of X. Then

(i) $\gamma_{g\alpha g}\text{Int}(A) \cup \gamma_{g\alpha g}\text{Int}(B) \subset \gamma_{g\alpha g}\text{Int}(A \cup B)$

(ii) $\gamma_{g\alpha g}\text{Int}(A \cap B) \subset \gamma_{g\alpha g}\text{Int}(A) \cap \gamma_{g\alpha g}\text{Int}(B)$

PROOF: Proof follows from lemma 2.6.34.

THEOREM 2.6.38: $\gamma_{g\alpha g}\text{Int}(A), \gamma_{g\alpha g}\text{Ext}(A)$ are disjoint where $A \subset X$ and hence for $x \in A$, $U \in \gamma_{g\alpha g}\text{-N}(x)$ & $\gamma_{g\alpha g}\text{Ext}(A)$ are disjoint

PROOF: $\gamma_{g\alpha g}\text{Cl}(A) \subset A$ and $\gamma_{g\alpha g}\text{Ext}(A) = \gamma_{g\alpha g}\text{Int}(A^c) \subset A^c$

$\therefore \gamma_{g\alpha g}\text{Int}(A) \cap \gamma_{g\alpha g}\text{Ext}(A) = \gamma_{g\alpha g}\text{Int}(A) \cap \gamma_{g\alpha g}\text{Int}(A^c) = \gamma_{g\alpha g}\text{Int}(A \cap A^c) = \gamma_{g\alpha g}\text{Int}(\emptyset) = \emptyset$. Thus $\gamma_{g\alpha g}\text{Int}(A)$ & $\gamma_{g\alpha g}\text{Ext}(A)$ are disjoint.

Now by Theorem 2.6.30 $\gamma_{g\alpha g}\mathrm{Int}(A)=\bigcup\{U\in\gamma_{g\alpha g}\text{-}N(x):U\subset A\}\Rightarrow$ U and $\gamma_{g\alpha g}\mathrm{Ext}(A)$ are disjoint. Hence the result.

THEOREM 2.6.39: For a subset A of a space X, the following statements hold:

1) $\gamma_{g\alpha g}b(A)\subset b(A)$ where b(A) denotes the boundary of A;

2) $A=\gamma_{g\alpha g}\mathrm{Int}(A)\cup b_{\alpha}(A)$;

3) $\gamma_{g\alpha g}\mathrm{Int}(A)\cap\gamma_{g\alpha g}b(A)=\emptyset$;

4) A is an $\gamma_{g\alpha g}$–open set if and only if $\gamma_{g\alpha g}b(A)=\emptyset$;

5) $\gamma_{g\alpha g}b(\gamma_{g\alpha g}\mathrm{Int}(A))=\emptyset$;

6) $\gamma_{g\alpha g}\mathrm{Int}(\gamma_{g\alpha g}b(A))=\emptyset$;

7) $\gamma_{g\alpha g}b(\gamma_{g\alpha g}b(A))=\gamma_{g\alpha g}b(A)$;

8) $\gamma_{g\alpha g}b(A))=A\cap\gamma_{g\alpha g}\mathrm{Cl}(X-\{A\}$

9) $\gamma_{g\alpha g}b(A)=\gamma_{g\alpha g}D(X-A)$

Proof: Easy to prove.

The converse of the theorem 2.6.39 (1) is not true in general

EXAMPLE 2.6.40: X = {a,b,c} τ= { Ø, {a}, {a,b}, {a,c}, X}. If A = {a,b}, then $\gamma_{g\alpha g}b(A)=\emptyset$ and b(A) = {b}. Hence $b(A)\not\subset\gamma_{g\alpha g}b(A)$

THEOREM 2.6.41: For a subset A of a space X, the following statements hold.

1) $\gamma_{g\alpha g}\mathrm{Fr}(A)\subset\mathrm{Fr}(A)$ where Fr(A) denotes the frontier of A;

2) $\gamma_{g\alpha g}\mathrm{Cl}(A)=\gamma_{g\alpha g}\mathrm{Int}(A)\cup\gamma_{g\alpha g}\mathrm{Fr}(A)$;

3) $\gamma_{g\alpha g}\mathrm{Int}(A)\cap\gamma_{g\alpha g}\mathrm{Fr}(A)=\emptyset$;

4) $\gamma_{g\alpha g}b(A)\subset\gamma_{g\alpha g}\mathrm{Fr}(A)$

5) $\gamma_{g\alpha g}\mathrm{Fr}(A)=\gamma_{g\alpha g}b(A)\cup\gamma_{g\alpha g}D(A)$;

6) A is an $\gamma_{g\alpha g}$–open set if and only if $\gamma_{g\alpha g}\mathrm{Fr}(A)=\gamma_{g\alpha g}D(A)$;

7) $\gamma_{g\alpha g}\mathrm{Fr}(A)=\gamma_{g\alpha g}\mathrm{Cl}(A)\cap\gamma_{g\alpha g}\mathrm{Cl}(X-A)$;

8) $\gamma_{g\alpha g}\mathrm{Fr}(A)=\gamma_{g\alpha g}\mathrm{Fr}(X-A)$;

9) $\gamma_{g\alpha g}Fr(A)$ is $\gamma_{g\alpha g}$-closed;

10) $\gamma_{g\alpha g}Fr(\gamma_{g\alpha g}Fr(A)) \subset \gamma_{g\alpha g}Fr(A)$;

11) $\gamma_{g\alpha g}Fr(\gamma_{g\alpha g}Int(A)) \subset \gamma_{g\alpha g}Fr(A)$;

12) $\gamma_{g\alpha g}Fr(\gamma_{g\alpha g}Cl(A)) \subset \gamma_{g\alpha g}Fr(A)$;

13) $\gamma_{g\alpha g}Int(A) = A - \gamma_{g\alpha g}Fr(A)$;

PROOF: Proof is straight forward

The converse of (1) and (4) of Theorem 2.6.41 are not true in general, as shown by the following example.

EXAMPLE 2.6.42: Let $X = \{a,b,c\}$ & $\tau = \{X, \emptyset, \{a\}, \{a,c\}\}$ be a topology on X. Then $\gamma_{g\alpha g}O(X) = \{X, \emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}\}$. If $A = \{c\}$, $B = \{b,c\}$, then $Fr(A) = \{b, c\} \not\subset \{c\} = \gamma_{g\alpha g}Fr(A)$, $\gamma_{g\alpha g}Fr(B) = \{b,c\} \not\subset \gamma_{g\alpha g}b\,(B) = \{c\}$.

THEOREM 2.6.43: Let (X,τ) be a topological space and A be the subset of X. Then A is an $\gamma_{g\alpha g}$-open in X if and only if and only if $F \subset \gamma_{g\alpha g}$-Int(A), where F is $\gamma_{g\alpha g}$-closed and $F \subset A$.

PROOF: Proof is easy.

CHAPTER 3 CONTINOUS FUNCTIONS

3.1. INTRODUCTION

Mashour et.al [32] in 1983, introduced and investigated, α-continuity in topological space. In this chapter we introduce and investigate the concept of (p,α)-continuity, (s,α)-continuity, αg-continuity, (α,αg)-continuity, contra αg-continuity. Some characterizations and basic properties of the new type of functions are obtained.

3.2 $\gamma_{g\alpha g}$ CONTINUOUS FUNCTIONS, (α,$\gamma_{g\alpha g}$)-CONTINUOUS & $\gamma_{g\alpha g}$-IRRESOLUTE FUNCTIONS

DEFINITION 3.2.1: A function f:(X,τ)→(Y,σ) is said to be **$\gamma_{g\alpha g}$-continuous** if the inverse image of every open set in Y is $\gamma_{g\alpha g}$-open in X.

EXAMPLE 3.2.2: Let X = Y = {a,b,c}. τ = {∅,{a},{b},{a,b},X}, σ = {∅, {a},{a,b},Y}. Then $\gamma_{g\alpha g}$C(X) ={∅,{c},{b,c},{a,c},X}, Let f be the identity map then f is $\gamma_{g\alpha g}$-continuous.

THEOREM 3.2.3: A function f: (X,τ) → (Y,σ) is $\gamma_{g\alpha g}$-continuous if and only if f^{-1}(V) is $\gamma_{g\alpha g}$-closed in X for every closed set in Y.

PROOF: Let f: (X,τ) → (Y,σ) is $\gamma_{g\alpha g}$-continuous and V be an closed set in Y. Then V^c is open in Y and since f is $\gamma_{g\alpha g}$-continuous, $f^{-1}(V^c)$ is $\gamma_{g\alpha g}$-open set in X. But $f^{-1}(V^c) = [f^{-1}(V)]^c$. So f^{-1}(V) is $\gamma_{g\alpha g}$-closed in X.
Conversely, assume that f^{-1}(V) is $\gamma_{g\alpha g}$-close in X for each closed set V in Y. Let F be a open set in Y. Then F^c is closed in Y and by assumption

$f^{-1}(F^c)$ is $\gamma_{g\alpha g}$-closed in X. Since $f^{-1}(F^c) = [f^{-1}(F)]^c$, we have $f^{-1}(F)$ is $\gamma_{g\alpha g}$-open set in X and so f is $\gamma_{g\alpha g}$-continuous.

THEOREM 3.2.4: If f: X→Y is $\gamma_{g\alpha g}$-continuous and g: Y→Z is continuous then their composition gof : X→Z is $\gamma_{g\alpha g}$-continuous.

PROOF: Let F be any open set in Z. Since g is continuous, $g^{-1}(F)$ is open set in Y. Since f is $\gamma_{g\alpha g}$-continuous and $g^{-1}(F)$ is open in $Y \Rightarrow f^{-1}(g^{-1}(F)) = (gof)^{-1}(F)$ is $\gamma_{g\alpha g}$-open set in X and so gof is $\gamma_{g\alpha g}$-continuous.

DEFINITION 3.2.5: A function f: X→Y is called **($\alpha,\gamma_{g\alpha g}$)-continuous** if $f^{-1}(V)$ is $\gamma_{g\alpha g}$-closed set in X for every α-closed set V of Y.

EXAMPLE 3.2.6: Let X = Y = {a,b,c}. τ = {∅,{a,b},X}, σ = {∅, {a}, {b},{a,b},Y}. Then $\gamma_{g\alpha g}C(X)$={∅,{a},{b},{c},{b,c},{a,c},X} and σC(X) ={∅,{c},{b,c},{a,c},X}. Let f be the identity map then f is ($\alpha,\gamma_{g\alpha g}$)-continuous.

THEOREM 3.2.7: If f: (X,τ) → (Y,σ) is α-irresolute then it is ($\alpha,\gamma_{g\alpha g}$)-continuous but not conversely.

PROOF: Let V be any α-closed in Y. since f is α-irresolute, $f^{-1}(V)$ is α-closed set in X. Every α-closed is $\gamma_{g\alpha g}$-closed in X. So $f^{-1}(V)$ is $\gamma_{g\alpha g}$-closed set in X. So f is ($\alpha,\gamma_{g\alpha g}$)-continuous.

However the converse of the above theorem need not be true.

EXAMPLE 3.2.8: Let X=Y={a,b,c}. τ ={∅,{a,b},X}, σ = {∅, {a}, {b}, {a,b},{a,c}Y}. Then αC(X)={∅,{c},X} & $\gamma_{g\alpha g}C(X)$ ={∅,{a}, {b}, {c}, {b,c},{a,c},X} and σC(X) ={∅,{b}, {c},{b,c}, {a,c}, X}. Let f be the identity map then f is $(\alpha,\gamma_{g\alpha g})$-continuous but not α-irresolute because the inverse image of α-closed set {b} in Y is {b}, which is not α-closed in X.

REMARK 3.2.9: The composition of two $(\alpha,\gamma_{g\alpha g})$-continuous functions need not be $(\alpha,\gamma_{g\alpha g})$-continuous.

EXAMPLE 3.2.10: Let X = Y =Z = {a,b,c}, τ ={∅,{a},{b},{a,b},{a,c}, X}, σ = {∅, {a}, {b}, {a,b},Y} and η ={∅,{a},Z}. Let f: (X,τ) → (Y,σ) be an identity map and g:(Y,σ)→(Z,η) defined by g(a)=c, g(b)= b, g(c)=a. Then f and g are $(\alpha,\gamma_{g\alpha g})$-continuous but their composition function gof: (X,τ) →(Z,η) is not $(\alpha,\gamma_{g\alpha g})$-continuous because V ={c} is α-closed set in Z, but $(gof)^{-1}(V) = f^{-1}g^{-1}(V) = f^{-1}(g^{-1}(\{c\}) = f^{-1}(\{a\}) = \{a\}$ is not $\gamma_{g\alpha g}$-closed in X.

DEFINITION 3.2.11: A function f:X→Y is called **$\gamma_{g\alpha g}$-irresolute** function if the inverse image of every $\gamma_{g\alpha g}$-closed set in Y is $\gamma_{g\alpha g}$-closed set in X.

REMARK 3.2.12: The following example shows that the notions of irresolute functions and $\gamma_{g\alpha g}$-irresolute functions are independent.

EXAMPLE 3.2.13: Let X = Y = {a,b,c}. τ = {∅, {a}, {a,b}, X}, σ ={∅, {a},{b},{a,b},Y}. Then $\gamma_{g\alpha g}C(X)$ = {∅, {b}, {c}, {b,c}, {a,c}, X} and $\gamma_{g\alpha g}C(Y)$ ={∅,{c},{b,c},{a,c},Y}. Then f : X→Y defined by f(a) = a, f(b) = b, f(c) =c is $\gamma_{g\alpha g}$-irresolute function, but it is not irresolute. Since {b.c} is semi-open and $\gamma_{g\alpha g}$-closed in Y but f^{-1}\{b,c\} ={b,c} is not semi-open in X where as {b,c} is $\gamma_{g\alpha g}$ -closed in X.

EXAMPLE 3.2.14: Let X = Y = {a,b,c}. τ ={∅, {a},{b},{a,b},X}, σ = {∅,{a,b},Y}. Then $\gamma_{g\alpha g}$C(X) ={∅,{c},{b,c},{a,c}, X}. $\gamma_{g\alpha g}$C(Y) ={∅,{a}, {b},{c},{b,c},{a,c},Y} and Then f : X→Y defined by f(a) = a, f(b) = b, f(c) =c is not $\gamma_{g\alpha g}$-irresolute function, but it is irresolute.

THEOREM 3.2.15: A function f: (X,τ) → (Y,σ) is $\gamma_{g\alpha g}$-irresolute function if and only the inverse image of every $\gamma_{g\alpha g}$-open set in Y is $\gamma_{g\alpha g}$-open set in X.

PROOF: Let f: (X,τ) → (Y,σ) be a $\gamma_{g\alpha g}$-irresolute function and U be an $\gamma_{g\alpha g}$-open set in Y. Then U^c is $\gamma_{g\alpha g}$-closed set in Y and since f is $\gamma_{g\alpha g}$-irresolute, $f^{-1}(U^c)$ is $\gamma_{g\alpha g}$-closed set in X. But $f^{-1}(U^c)$ = $[f^{-1}(U^c)]^c$ and so $f^{-1}(U)$ is $\gamma_{g\alpha g}$-open in X.

Conversely, assume that $f^{-1}(U)$ is $\gamma_{g\alpha g}$-open in X for each $\gamma_{g\alpha g}$-open set U in Y. Let F be $\gamma_{g\alpha g}$-closed set in Y. Then F^c is $\gamma_{g\alpha g}$-open in Y and by assumption $f^{-1}(F^c)$ is $\gamma_{g\alpha g}$-open in X. Since $f^{-1}(F^c)$ = $[f^{-1}(F)]^c$, we have $f^{-1}(F)$ is $\gamma_{g\alpha g}$-closed set in X and so f is $\gamma_{g\alpha g}$-irresolute.

THEOREM 3.2.16: If a function f: (X,τ) → (Y,σ) is $\gamma_{g\alpha g}$-irresolute then it is $\gamma_{g\alpha g}$-continuous.

PROOF: Proof is trivial.

REMARK 3.2.17: The converse of the theorem 3.6.16 need not be true, which is illustrated in the following theorem.

EXAMPLE 3.2.18: Let X = Y = {a,b,c}. τ ={∅,{a},{b},{a,b},{a,c}, X}, σ = {∅,{a},{a,c},Y}. Then $\gamma_{g\alpha g}$C(X) ={∅,{b},{c},{a,c},{b,c},X}, αC(Y) ={∅, {b,c},{b}, X} and $\gamma_{g\alpha g}$C(Y) ={X,∅,{b},{c},{a,b},{b,c}}. Then the identity function f:X →Y is $\gamma_{g\alpha g}$-continuous, since inverse image of every closed set in Y is $\gamma_{g\alpha g}$-closed set in X. But f is not $\gamma_{g\alpha g}$-irresolute since the inverse image of the $\gamma_{g\alpha g}$-closed set {a,b} is {a,b}, which is not a $\gamma_{g\alpha g}$-closed set.

THEOREM 3.2.19: Let X, Y, Z be topological spaces and $f:(X,\tau) \rightarrow (Y,\sigma)$ and $g: (Y,\sigma) \rightarrow (Z,\eta)$ be two functions such that

(i) f is $\gamma_{g\alpha g}$-irresolute and g is $\gamma_{g\alpha g}$-continuous function or

(ii) f is $\gamma_{g\alpha g}$-continuous and g is continuous function

Then their composition gof: $(X,\tau) \rightarrow (Z,\eta)$ is $\gamma_{g\alpha g}$-continuous function.

PROOF: (i) Let V be an open set in Z. Then $g^{-1}(V)$ is $\gamma_{g\alpha g}$-open. Since f is $\gamma_{g\alpha g}$-irresolute, $f^{-1}[g^{-1}(V)]$ is $\gamma_{g\alpha g}$-open. But $f^{-1}[g^{-1}(V)] =(gof)^{-1}(V)$. So gof is $\gamma_{g\alpha g}$-continuous.

Let U be a closed set in Z. Then $g^{-1}(U)$ be a closed set in Y because g is continuous. Since f is $\gamma_{g\alpha g}$-continuous, $f^{-1}[g^{-1}(U)]$ is $\gamma_{g\alpha g}$-clossed set in X. But $f^{-1}[g^{-1}(U)] = (gof)^{-1}(U)$. So gof is $\gamma_{g\alpha g}$-continuous.

THEOREM 3.2.20: Let X, Y, Z be topological spaces and $f:(X,\tau) \rightarrow (Y,\sigma)$ and $g: (Y,\sigma) \rightarrow (Z,\eta)$ be two $\gamma_{g\alpha g}$-irresolute functions. Then their composition gof: $(X,\tau) \rightarrow (Z,\eta)$ is $\gamma_{g\alpha g}$-irresolute function.

PROOF: Proof is trevial.

CHAPTER 4 SEPARATION AXIOMS

4.1 INTRODUCTION

α-separation axioms play a very important role in general topology. Indeed there are some research papers which deal with different α-separation axioms and also many topologists worldwide are doing research in this area. It is the aim of this part to offer some new types of α-separation axioms by using α-open sets. Separation axioms on the new set $\gamma_{g\alpha g}$-open sets are also introduced. As α-separation axioms play a very important role in general topology, $\gamma_{g\alpha g}$– separation axioms can also give their supporting contribution. It is the aim of this part to offer some new types of $\gamma_{g\alpha g}$ -separation axioms by using $\gamma_{g\alpha g}$ -open sets.

4.2 SOME BASIC CHARACTERIZATIONS OF $\gamma_{g\alpha g}$-SEPARATION AXIOMS

DEFINITION 4.2.1:A space X is said to be **$\gamma_{g\alpha g}$ -T_0** if for each pair of distinct points x & y in X, there exists a $\gamma_{g\alpha g}$-open set of X containing one point but not the other.

THEOREM 4.2.2: A topological space X is $\gamma_{g\alpha g}$ -T_0 space if and only if $\gamma_{g\alpha g}$-closure of distinct points are distinct.

PROOF: Let x, y$\in$X with x $\neq$ y and (X,τ) is $\gamma_{g\alpha g}$ -T_0 space. We will show that $\gamma_{g\alpha g}Cl(\{x\}) \neq \gamma_{g\alpha g}Cl(\{y\})$. Since (X,$\tau$) is $\gamma_{g\alpha g}$ -T_0, there exists $\gamma_{g\alpha g}$-open set G such that x$\in$G but y$\notin$G. Also x$\notin$X-G and y$\in$X-G, where X-G is $\gamma_{g\alpha g}$-closed set in (X,τ). Now {y} is the intersection of all $\gamma_{g\alpha g}$-closed sets which contain y. Hence $y\in\gamma_{g\alpha g}Cl(\{y\})$ but $x\notin \gamma_{g\alpha g}Cl(\{y\})$ as $x\notin X-G$. Therefore that $\gamma_{g\alpha g}Cl(\{x\}) \neq \gamma_{g\alpha g}Cl(\{y\})$.

Conversely, for any pair of distinct points $x,y\in X$ and $\gamma_{g\alpha g}Cl(\{x\}) \neq \gamma_{g\alpha g}Cl(\{y\})$. Then there exists at least one point $z\in X$ such that $z\in \gamma_{g\alpha g}Cl(\{x\})$ but $z\notin \gamma_{g\alpha g}Cl(\{y\})$. We claim that $x\notin \gamma_{g\alpha g}Cl(\{y\})$ because if $x\in\gamma_{g\alpha g}Cl(\{y\})$, then $\{x\}\subset \gamma_{g\alpha g}Cl(\{y\})\Rightarrow \gamma_{g\alpha g}Cl(\{x\})\subset \gamma_{g\alpha g}Cl(\{y\})$ (Using theorem 2.6.4). So $z\in \gamma_{g\alpha g}Cl(\{y\})$ which is contradiction. Hence $x\notin \gamma_{g\alpha g}Cl(\{y\})$. Now $x\notin \gamma_{g\alpha g}Cl(\{y\})\Rightarrow x\in X- \gamma_{g\alpha g}Cl(\{y\})$, which is an $\gamma_{g\alpha g}$-open set in (X,τ) containing x but not y. Hence (X,τ) is $\gamma_{g\alpha g}$ -T_0 space.

THEOREM 4.2.3: Every subspace of $\gamma_{g\alpha g}$ -T_0 space is $\gamma_{g\alpha g}$ -T_0 space. In other words the property of being a $\gamma_{g\alpha g}$ -T_0 space is a hereditary property.

PROOF : Let (X,τ) be a topological space. (Y,τ*) be a subspace of (X,τ), where τ* is a relative topology. Let y_1 & y_2 be two distinct points of Y and $Y\subset X$, therefore these two are distinct points of X. Since (X,τ) is a $\gamma_{g\alpha g}$ -T_0 space, there exists $\gamma_{g\alpha g}$ –open set G such that $y_1\in G$ and $y_2\notin G$. Then by definition, $G\cap Y$ is $\gamma_{g\alpha g}$ –open set in (Y, τ*) which contains y_1 but not y_2. hence (Y,τ*) is a $\gamma_{g\alpha g}$ -T_0 space.

DEFINITION 4.2.4 : A space X is said to be $\boldsymbol{\gamma_{g\alpha g}}$**-$\boldsymbol{T_1}$** if for each pair of distinct points x and y of X , there exist $\gamma_{g\alpha g}$-open sets U and V containing x and y respectively such that $y\notin U$ and $x\notin V$.

EXAMPLE 4.2.5: (1) Let X = {a,b,c,d}, τ ={Ø,{a,d},{a,b,c},X}
$\gamma_{g\alpha g}O(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{a,b,c},{a,b,d},{a,c,d},
{b,c,d},X}. Then (X,τ) is $\gamma_{g\alpha g}$-T_1.

(2) Let X={a,b,c} τ ={Ø,{a},{b},{a,b},X}. Then $\gamma_{g\alpha g}O(X)$ ={Ø,{a},{b}, {a,b}, X}. Then (X,τ) is not $\gamma_{g\alpha g}$-T_1 since for the elements a and c of X, there does not exist $\gamma_{g\alpha g}$-open sets U and V containing a and c respectively such that $c \notin U$ and $a \notin V$.

REMARK 4.2.6:(i) Every α-T_1 space is $\gamma_{g\alpha g}$-T_1. Since every α-open set is $\gamma_{g\alpha g}$-open set. (ii) Every $\gamma_{g\alpha g}$-T_1 space is $\gamma_{g\alpha g}$-T_0.

The converse of the above remark need not be true as seen from following examples.

EXAMPLE 4.2.7: **(1)** Let X = {a,b,c,d}, τ ={Ø,{a,d},{a,b,c},X}. Then (X,τ) is $\gamma_{g\alpha g}$-T_1 but not α-T_1 space.

(2) Let X={a,b,c} τ ={Ø,{a},{a,b},{a,c},X}. Then $\gamma_{g\alpha g}O(X)$ = {Ø, {a}, {a,b},{a,c}, X}. Then (X,τ) is $\gamma_{g\alpha g}$-T_0 but not $\gamma_{g\alpha g}$-T_1 since for the elements a and c of X, there does not exist $\gamma_{g\alpha g}$-open sets U and V containing a and c respectively such that $c \notin U$ and $a \notin V$.

PROPOSITION 4.2.8: If A is $\gamma_{g\alpha g}$–closed, then $C\gamma_{g\alpha g}(A)$-A does not contain non-empty closed set.

PROOF: Let F be a closed subset of $C\gamma_{g\alpha g}(A)$-A. Then $C\gamma_{g\alpha g}(A) \subset F^c$. We have $F \subset \left[C\gamma_{g\alpha g}(A)\right]^c \cap C\gamma_{g\alpha g}(A) = Ø$ and hence F = Ø

PROPOSITION 4.2.9:(i) For each $x \in X$, {x} is closed or its complement X-{x} is $\gamma_{g\alpha g}$– closed in (X, τ)

(ii) For each $x \in X$, {x} is α-closed or its complement X-{x} is $\gamma_{g\alpha g}$– closed in (X, τ)

PROOF: (i) Suppose {x} is not closed. Then its complement X-{x} is not open. X is the only open set containing X-{x}. We have $C\gamma_{g\alpha g}$(X- {x}) $\subset$ X holds. This implies X-{x} is $\gamma_{g\alpha g}$- closed in (X, τ).

(iii) The proof is similar to (i).

(iv)

THEOREM 4.2.10: A topological space (X, τ) is $\gamma_{g\alpha g}T_1$ if and only if the singletons are $\gamma_{g\alpha g}$ -closed.

PROOF: Let (X, τ) be a $\gamma_{g\alpha g}T_1$ and x any point of X. suppose $y \in \{x\}^c$. Then $x \neq y$ and so there exists a $\gamma_{g\alpha g}$-open set U_y such that $y \in U_y$ but $x \notin Uy$. Consequently $y \in U_y \subset \{x\}^c$ i.e., $\{x\}^c = \cup\{U_y \mid y \in \{x\}^c\}$ which is α-open.

Conversely suppose {p} is $\gamma_{g\alpha g}$-closed for every $p \in X$. Let x, y $\in$ X with $x \neq y$. Now $x \neq y$ implies $y \in \{x\}^c$. Hence $\{x\}^c$ is $\gamma_{g\alpha g}$-open set containing y but not x. Similarly $\{y\}^c$ is a $\gamma_{g\alpha g}$-open containing x but not y. Accordingly X is a $\gamma_{g\alpha g}$-T_1 space.

THEOREM 4.2.11: The product space of two $\gamma_{g\alpha g}$-T_0 spaces is an $\gamma_{g\alpha g}$-T_0 space.

PROOF: Let (X,τ_1) and (Y,τ_2) be two topological spaces and $(X\times Y,\tau)$ be their product space. Let x and y be two distinct points of X. Then (X,τ_1) is $\gamma_{g\alpha g}$-T_0 space iff there exist an $\gamma_{g\alpha g}$-open set G such that it contains only one of these to and not the other. We claim that $(X\times Y,\tau)$ $\gamma_{g\alpha g}$-T_0 space. Let (x_1,y_1) and (x_2,y_2) be any two distinct points of $X\times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ and (X,τ_1) being $\gamma_{g\alpha g}$-T_0 space, there exist an $\gamma_{g\alpha g}$-open set G in (X,τ_1) such that $x_1 \in G$ and $x_2 \notin G$. Then $G\times Y$ is an $\gamma_{g\alpha g}$-open set in $(X\times Y,\tau)$ containing (x_1,y_1) but not containing (x_2,y_2). Similarly if $y_1 \neq y_2$ and (Y,τ_2) being $\gamma_{g\alpha g}$-T_0 space, there exist an $\gamma_{g\alpha g}$-open set H in

(Y,τ_2) such that $y_1 \in H$ and $y_2 \notin H$. Then $X \times H$ is an $\gamma_{g\alpha g}$- open set in $(X \times Y,\tau)$ containing (x_1,y_1) but not containing (x_2,y_2). Hence corresponding to distinct points of $X \times Y$, there exist an $\gamma_{g\alpha g}$-open set containing one but not the other so that $(X \times Y,\tau)$ is also $\gamma_{g\alpha g}$-T_0 space.

DEFINITION 4.2.12 : A space X is said to be **$\gamma_{g\alpha g}$ -T_2** if for each pair of distinct points x and y in X , there exist disjoint $\gamma_{g\alpha g}$-open sets U and V in X such that $x \in U$ and $y \in V$.

EXAMPLE 4.2.13: Let X = {a,b,c,d}, τ ={Ø,{a,b},{a,b,c},X}

$\gamma_{g\alpha g}O(X)$ ={Ø,{a},{b},{c},{a,b},{a,c},{a,d},{b,c},{b,d},{a,b,c}, {a,b,d},{a,c,d},{b,c,d},X}. Then X is $\gamma_{g\alpha g}$ –T_2.

REMARK 4.2.14: (i) Every α-T_2 space is $\gamma_{g\alpha g}$ -T_2 space (ii) Every $\gamma_{g\alpha g}$ -T_2 space is $\gamma_{g\alpha g}$ –T_1 space.

DEFINITION 4.2.15: A topological space (X, τ) is **$\gamma_{g\alpha g}$ -symmetric** if for x and y in X, $x \in \gamma_{g\alpha g}$ Cl({y}) implies $y \in \gamma_{g\alpha g}$Cl({x}).

THEOREM 4.2.16: A topological space (X, τ) is $\gamma_{g\alpha g}$-symmetric if and only if {x} is $\gamma_{g\alpha g}$ -closed for each $x \in X$.

PROOF: Assume that $x \in \gamma_{g\alpha g}$Cl({y}) but $y \notin \gamma_{g\alpha g}$Cl({x}). This means that $\gamma_{g\alpha g}$Cl$(\{x\})^c$ contains y. This implies that $\gamma_{g\alpha g}$Cl({y}) is a subset of $\gamma_{g\alpha g}$Cl$(\{x\})^c$. But then $\gamma_{g\alpha g}$Cl$(\{x\})^c$ contains x which is a contadiction.

Conversely, suppose that $\{x\} \subset U$, where U is $\gamma_{g\alpha g}$-open set but $\gamma_{g\alpha g}$Cl({x}) is not a subset of U. This means that $\gamma_{g\alpha g}$Cl ({x}) and U^c are not disjoint. Let y belongs to their intersection. Now we have $x \in$

$\gamma_{g\alpha g}Cl(\{y\})$ which is a subset of U^c and $x \notin U$. But this is a contradiction to our assumption. Hence the result.

COROLLARY 4.2.17: If a topological space (X, τ) is $\gamma_{g\alpha g}$-T_1 space then it is $\gamma_{g\alpha g}$-symmetric.

PROOF: In a $\gamma_{g\alpha g}$-T_1 space, singleton sets are $\gamma_{g\alpha g}$-closed (Theorem 4.2.10). So by Theorem 4.2.16, the space is $\gamma_{g\alpha g}$-symmetric.

COROLLARY 4.2.18: For a topological space (X, τ) the following are equivalent

(i) (X, τ) is $\gamma_{g\alpha g}$ -symmetric and $\gamma_{g\alpha g}$ -T_0

(ii) (X, τ) is $\gamma_{g\alpha g}$ -T_1.

PROOF: (i) $\Rightarrow$(ii): Let $x \neq y$ and by $\gamma_{g\alpha g}$-T_0, we may assume that $x \in G_1 \subset \{y\}^c$ for $G_1 \in \gamma_{g\alpha g}O(X,\tau)$. Then $x \notin \gamma_{g\alpha g}Cl\{y\}$ and $y \notin \gamma_{g\alpha g}Cl\{x\}$. There exists a $G_2 \in \gamma_{g\alpha g}O(X,\tau)$ such that $y \in G_2 \subset \{x\}^c$ and (X, τ) is $\gamma_{g\alpha g}$-T_1 space.

(ii) $\Rightarrow$(i) By corollary 4.2.19, (X, τ) is $\gamma_{g\alpha g}$-symmetric. Also it is trivial that (X, τ) is $\gamma_{g\alpha g}$-$T_1 \Rightarrow (X, \tau)$ is $\gamma_{g\alpha g}$-T_0.

DEFINITION 4.2.19: A space (X,τ) is said to be **$\gamma_{g\alpha g}$-T space** if every $\gamma_{g\alpha g}$-closed set in it is closed set in X.

EXAMPLE 4.2.20: (a) Let $X = \{a,b,c\}$, $\tau = \{\emptyset,\{a\},\{b,c\},X\}$. Then (X,τ) is $\gamma_{g\alpha g}$-T space.

(b) Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. Then $\{c\}$ is $\gamma_{g\alpha g}$-closed but not closed in (X,τ).

THEOREM 4.2.21: Let X and Z be topological spaces and Y be $\gamma_{g\alpha g}$-T space. Let f:X →Y and g:Y→Z be two (α,$\gamma_{g\alpha g}$)-continuous. Then the composite function gof: X →Z of the (α,$\gamma_{g\alpha g}$)-continuous functions.

PROOF: Let F be any α-closed in Z . As g is (α,$\gamma_{g\alpha g}$)-continuous, g^{-1}(F) is $\gamma_{g\alpha g}$-closed set in Y. Since Y is $\gamma_{g\alpha g}$-T space every $\gamma_{g\alpha g}$-closed set is closed in Y. Hence g^{-1}(F) is closed in Y. Since every closed set is α-closed . So g^{-1}(F) is α-closed in Y. Now since f is (α,$\gamma_{g\alpha g}$)-continuous and g^{-1}(F) is α-closed in Y, f^{-1}(g^{-1}(F)) is $\gamma_{g\alpha g}$-closed in X. But f^{-1}(g^{-1}(F)) = $(gof)^{-1}$(F) and gof is (α,$\gamma_{g\alpha g}$)-continuous.

THEOREM 4.2.22: Let (X,τ) be any topological space, (Y,σ) be a $\gamma_{g\alpha g}$-T space and f : (X,τ)→ (Y,σ) be a function. Then the following are equivalent: (i) f is $\gamma_{g\alpha g}$-irresolute (ii) f is $\gamma_{g\alpha g}$-continuous.

PROOF: Every $\gamma_{g\alpha g}$-irresolute function is $\gamma_{g\alpha g}$-continuous. Hence (i) ⇒ (ii). Let F be an $\gamma_{g\alpha g}$-closed set in (Y,σ). Since (Y,σ) is a $\gamma_{g\alpha g}$-T space, F is closed set in X. Therefore f is $\gamma_{g\alpha g}$-irresolute. Hence (ii) ⇒ (i).

CHAPTER 5 OPEN FUNCTIONS

5.1. INTRODUCTION

In this chapter, we define and study some basic properties of α-open (resp. α-closed) functions, (p, α) – openness, (p, α) – closedness , αg-openness, αg-closedness. Some characterizations and basic properties of these new type of functions are obtained.

5.2. $\gamma_{g\alpha g}$–OPEN FUNCTIONS AND $\gamma_{g\alpha g}$–CLOSED FUNCTIONS

In this section, we introduce and study $\gamma_{g\alpha g}$ –open functions and $\gamma_{g\alpha g}$ –closed functions in topological spaces and study some of their properties.

DEFINITION 5.2.1: A function f: X→Y is called **$\gamma_{g\alpha g}$–open** if f(V) is $\gamma_{g\alpha g}$–open in Y for every open set V in X. or A function f: X→Y is called $\gamma_{g\alpha g}$–closed if f(V) is $\gamma_{g\alpha g}$–closed in Y for every closed set V in X.

DEFINITION 5.2.2: A function f:X→Y is called $\gamma_{g\alpha g}$-irresolute function if the inverse image of every $\gamma_{g\alpha g}$-closed set in Y is $\gamma_{g\alpha g}$-closed set in X.

THEOREM 5.2.3: A surjective function f: X→Y is called $\gamma_{g\alpha g}$–closed if and only if for each subset B of Y and for each open set U in X containing $f^{-1}(B)$, there is an $\gamma_{g\alpha g}$–open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

PROOF: Necessity: Assume that f is $\gamma_{g\alpha g}$–closed. Let B be a subset of Y and U be an open set of X containing $f^{-1}(B)$. Put V= Y-f(U^c). Then since f is $\gamma_{g\alpha g}$–closed set, V is $\gamma_{g\alpha g}$–open set in Y, $B \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency: Suppose for each subset B of X and for each open set U in X containing B, there is an $\gamma_{g\alpha g}$–open set V of Y such that f(B) $\subset$ V. Let F be a closed set in X. Put B = Y –f(F). Then $f^{-1}(V) \subset F^c$ and F^c is an open set in X. By hypothesis, there is an $\gamma_{g\alpha g}$–open set V of Y such that B = Y–f(F) $\subset$ V and $f^{-1}(V) \subset F^c$. So we obtain f(F) = V^c and hence f(F) is $\gamma_{g\alpha g}$–closed in Y. This shows that f is $\gamma_{g\alpha g}$–closed.

REMARK 5.2.4: Necessary of Theorem 5.2.3 is proved without assuming that f is surjective. Therefore, we can obtain the following corollary.

COROLLARY 5.2.5: If the function f: X$\rightarrow$Y is $\gamma_{g\alpha g}$–closed, then for any closed set F of Y and for any open set U containing $f^{-1}(F)$, there exists a $\gamma_{g\alpha g}$–open set V of Y such that F$\subset$V and $f^{-1}(V) \subset U$.

PROOF: By Theorem 5.2.3, there exists a $\gamma_{g\alpha g}$–open set W of Y such that F$\subset$W and $f^{-1}(W) \subset U$. Since F is closed, we have F$\subset \gamma_{g\alpha g}$-Int(W). Put V=$\gamma_{g\alpha g}$-Int(W), then V$\in \gamma_{g\alpha g}O(Y)$, F$\subset$V, and $f^{-1}(V) \subset U$.

PROPOSITION 5.2.6: If f: X$\rightarrow$Y is $\gamma_{g\alpha g}$-irresolute and $\gamma_{g\alpha g}$-closed. Let A is $\gamma_{g\alpha g}$-closed in X, then f(A) is $\gamma_{g\alpha g}$-closed in Y.

PROOF: Let V be any $\gamma_{g\alpha g}$-open set of Y containing f(A). Then $f^{-1}(V)$ is $\gamma_{g\alpha g}$-open in X, since f is $\gamma_{g\alpha g}$-irresolute. i.e., A$\subset f^{-1}(V)$. Since A is $\gamma_{g\alpha g}$-closed in X, $\gamma_{g\alpha g}$-Cl(A) $\subset f^{-1}(V)$ and hence f(A) $\subset$f($\gamma_{g\alpha g}$-Cl(A)$\subset$V. Since f is $\gamma_{g\alpha g}$-closed and $\gamma_{g\alpha g}$-Cl(A) is $\gamma_{g\alpha g}$-closed in X, f($\gamma_{g\alpha g}$-Cl(A)) is $\gamma_{g\alpha g}$-closed in Y and hence $\gamma_{g\alpha g}$-Cl(f(A)) $\subset$ $\gamma_{g\alpha g}$-Cl(f($\gamma_{g\alpha g}$-Cl(A))) $\subset$V. This shows that f(A) is $\gamma_{g\alpha g}$-closed in Y.

PROPOSITION 5.2.7: If f: X→Y is an $\gamma_{g\alpha g}$-open, pre-irresolute bijection and B is $\gamma_{g\alpha g}$-closed in Y then $f^{-1}(B)$ is $\gamma_{g\alpha g}$-closed in X.

PROOF:(i) Let U be any $\gamma_{g\alpha g}$-open set of X containing $f^{-1}(B)$. Then $B \subset f(U)$ and f(U) is $\gamma_{g\alpha g}$-open in Y, since f is $\gamma_{g\alpha g}$-open. Since B is $\gamma_{g\alpha g}$-closed in Y, $\gamma_{g\alpha g}$-Cl(B)$\subset$f(U) and hence $f^{-1}(B) \subset f^{-1}(\gamma_{g\alpha g}\text{-Cl}(B)) \subset U$. Since f is pre-irresolute, $f^{-1}(\gamma_{g\alpha g}\text{-Cl}(B))$ is $\gamma_{g\alpha g}$-open in X and hence $\gamma_{g\alpha g}\text{-Cl}(f^{-1}(B)) \subset f^{-1}(\gamma_{g\alpha g}\text{-Cl}(B))$. This shows $f^{-1}(B)$ is $\gamma_{g\alpha g}$-closed in X.

REMARK 5.2.8: The composition of two $\gamma_{g\alpha g}$-closed function need not be $\gamma_{g\alpha g}$-closed. This is illustrated in the following example.

EXAMPLE 5.2.9: Let X = Y = Z = {a,b,c}, τ = {∅, {a},{b},{a,b},X}, σ ={∅, {a},{b},{a,b},{a,c},Y}, η ={∅,{a,b},Z}. Let f:(X,τ)→(Y,σ) be the identity function and define g: (Y,σ)→(Z,η) by g(a) = g(b) = b, g(c) =a. Then f and g are $\gamma_{g\alpha g}$-closed functions but their composition gof: (X,τ)→(Z,η) is not $\gamma_{g\alpha g}$-closed function since for the closed set {c} in (X,τ), gof({c})={a} is not $\gamma_{g\alpha g}$-closed in (Z,η).

THEOREM 5.2.10: Let f:(X,τ)→(Y,σ) and g: (Y,σ)→(Z,η) be $\gamma_{g\alpha g}$-closed functions and (Y,σ) be a $\gamma_{g\alpha g}$-T space. Then their composition gof is $\gamma_{g\alpha g}$-closed function.

PROOF: Let A be a closed set of (X,τ). Then by hypothesis f(A) is an $\gamma_{g\alpha g}$-closed set in (Y,σ). Since (Y,σ) is a $\gamma_{g\alpha g}$-T space, f(A) is closed in (Y,σ). Also by assumption g(f(A)) is $\gamma_{g\alpha g}$-closed map in (Z,η). Hence gof is $\gamma_{g\alpha g}$-closed function.

THEOREM 5.2.11: Let f: $(X,\tau)\to(Y,\sigma)$ and g: $(Y,\sigma)\to(Z,\eta)$ be two mappings such that their composition gof:$X\to Z$ is an $\gamma_{g\alpha g}$-closed mapping. Then the following statements are true.

(i) if f is continuous and surjective then g is $\gamma_{g\alpha g}$-closed function.

(ii) If g is irresolute and injective then g is $\gamma_{g\alpha g}$-closed function.

PROOF: (i)Let A be a closed set of (Y,σ). Then $f^{-1}(A)$ is closed set in (X,τ) as f is continuous. Since gof is a $\gamma_{g\alpha g}$-closed and f is surjective, gof$[f^{-1}(A)]$ = g(A) is $\gamma_{g\alpha g}$-closed in (Z,η). Hence g is $\gamma_{g\alpha g}$-closed function in X.

(ii) Let H be a closed set of (X,τ). Since gof is $\gamma_{g\alpha g}$-closed function, gof(H) is an $\gamma_{g\alpha g}$-closed set in (Z,η). Since g is $\gamma_{g\alpha g}$-irresolute and g is injective, $[g^{-1}(gof)](H)=g^{-1}(g(f(H))=f(H)$ is $\gamma_{g\alpha g}$-closed in (Y,σ). Thus f is an $\gamma_{g\alpha g}$-closed function in X.

PROPOSITION 5.2.12: For any bijection f:$(X,\tau)\to(Y,\sigma)$, the following statements are equivalent.

(i) inverse of f is $\gamma_{g\alpha g}$-continuous

(ii) f is an $\gamma_{g\alpha g}$-open function

(iii) f is an $\gamma_{g\alpha g}$-closed function

PROOF: (i)⇒(ii). Let U be an open set of (X,τ). By assumption $(f^{-1})^{-1}(U)$ = f(U) is $\gamma_{g\alpha g}$-open in (Y,σ) and so f is $\gamma_{g\alpha g}$-open.

(ii)⇒(iii). Let F be a closed set of (X,τ). Then F^c is an open in (X,τ). By assumption $f(F^c)$ is $\gamma_{g\alpha g}$-open in (Y,σ). Therefore f(F) is $\gamma_{g\alpha g}$-closed function in (Y,σ). Hence f is $\gamma_{g\alpha g}$-closed function.

(iii)⇒(i). Let F be a closed set in (X,τ). By assumption f(F) is $\gamma_{g\alpha g}$-closed in (Y,σ). But $f(F) = (f^{-1})^{-1}(F)$ and so f is $\gamma_{g\alpha g}$-continuous.

DEFINITION 5.2.13: A function f: $(X,\tau)\rightarrow(Y,\sigma)$ is **strongly $\gamma_{g\alpha g}$-open** if f(U) is $\gamma_{g\alpha g}$-open in Y for every $\gamma_{g\alpha g}$-open set U in X.

THEOREM 5.2.14: If is f : $(X,\tau)\rightarrow(Y,\sigma)$ bijective, strongly $\gamma_{g\alpha g}$-open and $(\alpha,\gamma_{g\alpha g})$-continuous, then f is $\gamma_{g\alpha g}$-irresolute.

PROOF: Let A be $\gamma_{g\alpha g}$-closed set in Y. Let U be any $\gamma_{g\alpha g}$-open set in X such that $A \subset f(U)$. Since A is $\gamma_{g\alpha g}$-closed set and f(U) is $\gamma_{g\alpha g}$-open in Y, $A = \gamma_{g\alpha g}Cl(A) \subset f(U)$ hold and $f^{-1}(\gamma_{g\alpha g}Cl(A)) \subset U$. Since f is $\gamma_{g\alpha g}$-continuous and $\gamma_{g\alpha g}Cl(A)$ is $\gamma_{g\alpha g}$-closed set in Y, $f^{-1}(\gamma_{g\alpha g}Cl(A))$ is $\gamma_{g\alpha g}$-closed in X. We have $\gamma_{g\alpha g}Cl[f^{-1}(\gamma_{g\alpha g}Cl(A))] \subset U$ and so $\gamma_{g\alpha g}Cl[f^{-1}(A)] \subset U$. So $f^{-1}(A)$ is $\gamma_{g\alpha g}$-closed in X and hence f is $\gamma_{g\alpha g}$-irresolute.

THEOREM 5.2.15: If f : $(X,\tau)\rightarrow(Y,\sigma)$ is bijective, α-closed set and $\gamma_{g\alpha g}$-closed set and $\gamma_{g\alpha g}$-irresolute functions, then the inverse function $f^{-1}: (Y,\sigma) \rightarrow(X,\tau)$ is $\gamma_{g\alpha g}$-irresolute.

PROOF: Let A be $\gamma_{g\alpha g}$-closed set in X. Let $(f^{-1})^{-1}(A) = f(A) \subset U$ where U is $\gamma_{g\alpha g}$-open in Y. Then $A\subset f^{-1}(U)$ holds. Since $f^{-1}(U)$ is $\gamma_{g\alpha g}$-open in X and A is $\gamma_{g\alpha g}$-closed set in X. $A = \gamma_{g\alpha g}Cl(A) \subset f^{-1}(U)$ and hence $f[\gamma_{g\alpha g}Cl(A)]\subset U$. Since f is α-closed and $\gamma_{g\alpha g}Cl(A)$ is α-closed set in X, $f[\gamma_{g\alpha g}Cl(A)]$ is closed set in Y. So $f[\gamma_{g\alpha g}Cl(A)]$ is $\gamma_{g\alpha g}$-closed set in Y. Therefore $\gamma_{g\alpha g}Cl[f(\gamma_{g\alpha g}Cl(A))] \subset U$ and hence $\gamma_{g\alpha g}Cl[f(A)]\subset U$. Thus f(A) is $\gamma_{g\alpha g}$-closed set in Y and so f^{-1} is $\gamma_{g\alpha g}$-irresolute.

CHAPTER VI SOME REGULARITY AND NORMALITY AXIOMS

6.1 INTRODUCTION

In this chapter we have given some properties of α-regular space & α-normality. After this we have given the properties of αg-regular space & αg-normality. Finally we have defined $\gamma_{g\alpha g}$-regular space, $\gamma_{g\alpha g}$-symmetric space & $\gamma_{g\alpha g}$-symmetric space and have given their properties.

6.2 $\gamma_{g\alpha g}$–NORMAL SPACES, $\gamma_{g\alpha g}$–REGULAR SPACES AND $\gamma_{g\alpha g}$-SYMMETRIC SPACES

DEFINITION 6.2.1: For every set $A \subset X$, we define the $\gamma_{g\alpha g}$-closure of A as the intersection of all $\gamma_{g\alpha g}$-closed sets containing A. In symbols $\gamma_{g\alpha g}\text{-Cl}(A) = \cap\{F : A \subset F$ where F is $\gamma_{g\alpha g}$-closed in (X,τ).

DEFINITION 6.2.2: A space (X,τ) is said to be $\gamma_{g\alpha g}$-normal if for any pair of disjoint $\gamma_{g\alpha g}$-closed A and B in X, there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

REMARK 6.2.3: It is obvious that $\gamma_{g\alpha g}$-normal space is normal. However the converse is not true.

EXAMPLE 6.2.4: Let $X = \{a,b,c,d\}$. $\tau = \{\varnothing,\{a,d\},\{b,c\},X\}$. Then (X,τ) is normal but not $\gamma_{g\alpha g}$-normal.

THEOREM 6.2.5: The following are equivalent for a space (X,τ).

(i) (X,τ) is normal

(ii) For any disjoint closed sets A and B, there exists disjoint $\gamma_{g\alpha g}$-open sets U,V such that $A\subset U$ & $B\subset V$.

(iii) For any closed set A and any open set V containing A, there exist $\gamma_{g\alpha g}$-open set U of X such that $A \subset U \subset \gamma_{g\alpha g}\text{-}Cl(U) \subset U$.

PROOF: (i)⇒(ii). The result follows from the result that every open set is $\gamma_{g\alpha g}$-open.

(ii) ⇒ (iii). Let A be a closed set and V be an open set containing A. Then A & X-V are disjoint closed sets. There exists disjoint $\gamma_{g\alpha g}$-open sets U and W such that $A\subset U$ and $X\text{-}V\subset W$. Since X-V is closed, it is $\gamma_{g\alpha g}$-closed set, we have $X\text{-}V\subset \gamma_{g\alpha g}\text{-}Int(W)$ and $U\cap \gamma_{g\alpha g}\text{-}Int(W) = \emptyset$. So we have $\gamma_{g\alpha g}\text{-}Cl(U) \cap \gamma_{g\alpha g}\text{-}Int(W) = \emptyset$ and hence $A \subset U \subset \gamma_{g\alpha g}\text{-}Cl(U) \subset X\text{-} \gamma_{g\alpha g}\text{-}Int(W) \subset V$.

(iii) ⇒ (ii). Let A and B be disjoint closed sets of X. then $A \subset X\text{-}B$ and X-B is open. There exists an $\gamma_{g\alpha g}$-open set G of X such that $A \subset G \subset \gamma_{g\alpha g}\text{-}Cl(G) \subset X\text{-}B$. Since A is closed, it is $\gamma_{g\alpha g}$-closed, we have $A \subset \gamma_{g\alpha g}\text{-}Int(G)$. Put $U = Int(Cl(Int(\gamma_{g\alpha g}\text{-}Int(G)))$ and $V = Int(Cl(Int(X\text{-} \gamma_{g\alpha g}\text{-}Cl(G)))$. Then U and V are disjoint open sets of X such that $A\subset U$ and $B\subset V$. Then (X,τ) is normal.

THEOREM 6.2.6: The following are equivalent for a space (X,τ).

(i) (X,τ) is α-normal

(ii) For any pair of disjoint closed sets A and B, there exists $\gamma_{g\alpha g}$-open sets U,V such that $A\subset U$ & $B\subset V$ and $U\cap V=\emptyset$

PROOF: (i) ⇒ (ii). Assume that (X,τ) is α-normal. Let A and B be disjoint closed subsets of X. By hypothesis, there exists disjoint α-open sets (and hence $\gamma_{g\alpha g}$-open sets) U and V such that $A\subset U$ & $B\subset V$ and $U\cap V=\emptyset$.

(ii)$\Rightarrow$ (i). Let A and B be closed subsets of X. Then by assumption, $A \subset G$, $B \subset H$ and $G \cap H = \varnothing$ where G and H are disjoint $\gamma_{g\alpha g}$-open sets. Since A and B are $\gamma_{g\alpha g}$-closed sets in X, by theorem 2.6.43, $A \subset \gamma_{g\alpha g}\text{-Int}(G)$, $B \subset \gamma_{g\alpha g}\text{-Int}(H)$ and $\gamma_{g\alpha g}\text{-Int}(G) \cap \gamma_{g\alpha g}\text{-Int}(H) = \varnothing$. Hence (X,τ) is α-normal.

THEOREM 6.2.7: If (X,τ) is α-normal and $F \cap A = \varnothing$, where F is closed and A is $\gamma_{g\alpha g}$-closed, then there exists disjoint α-open sets U & V such that $F \subset U$ and $A \subset V$.

PROOF: Since F is closed and $F \cap A = \varnothing$, we have $A \subset F^c$ and so $\gamma_{g\alpha g}\text{-Cl}(A) \subset F^c$. Thus $Cl(A) \cap F = \varnothing$. Since F and Cl(A) are closed and (X,τ) is α-normal, there exists α-open sets U and V such that $Cl(A) \subset U$ and $F \subset V$. i.e., $A \subset U$ and $F \subset V$.

THEOREM 6.2.8: If (X,τ) is α-normal, the following statements are true.

(i) For each closed set A and every $\gamma_{g\alpha g}$-open set B such that $A \subset B$, there exists an α-open set U such that $A \subset U \subset C_\alpha(A) \subset B$.

(ii) For every $\gamma_{g\alpha g}$-closed set a and every open set B containing A there exists an α-open set U such that $A \subset U \subset C_\alpha(U) \subset B$.

PROOF: (i) Let A be closed and B be an $\gamma_{g\alpha g}$-open set such that $A \subset B$. Then $A \cap B^c = \varnothing$ where A is closed and B^c is $\gamma_{g\alpha g}$-closed. Therefore by theorem 6.2.7, there exists α-open sets U & V such that $A \subset U$, $B^c \subset V$ and $U \cap V = \varnothing$. Thus $A \subset U \subset V^c \subset B$. Since Vc is α-closed, $C_\alpha(U) \subset V^c$ and so $A \subset U \subset C_\alpha(U) \subset B$.

(ii) Let A be an $\gamma_{g\alpha g}$-closed set and B be an open set such that $A \subset B$. Then $B^c \subset A^c$. Since (X,τ) is α-normal and A^c is $\gamma_{g\alpha g}$-open set containing the closed set B^c, we have by (i), there exists an α-open set G such that $B^c \subset G$

$C_\alpha(G) \subset A^c$. Thus $A \subset C_\alpha(G))^c \subset G^c \subset B$. Let $U = C_\alpha(G))^c$. Then U is an α-open and $A \subset U \subset C_\alpha(U) \subset B$.

THEOREM 6.2.9: The following are equivalent

(i) (X,τ) is $\gamma_{g\alpha g}$-normal.

(ii) For each $\gamma_{g\alpha g}$-closed set A and for each $\gamma_{g\alpha g}$-open set U containing A, there exists an open set V containing A such that $Cl(V) \subset U$.

(iii) For each pair of disjoint $\gamma_{g\alpha g}$-closed sets A and B in (X,τ) there exists an open set U containing A such that $C_\alpha(U) \cap B = \emptyset$.

(iv) For each pair of disjoint $\gamma_{g\alpha g}$-closed sets A and B in (X,τ) there exists an open set U containing A and on open set V containing B such that $C_\alpha(U) \cap C_\alpha(V) = \emptyset$.

PROOF: (i) $\Rightarrow$ (ii). Let A be an $\gamma_{g\alpha g}$-closed set and U be an $\gamma_{g\alpha g}$-open set such that $A \subset U$. Then $A \cap U^c = \emptyset$. Since (X,τ) is $\gamma_{g\alpha g}$-normal, there exist open sets V & W such that $A \subset V$, $U^c \subset W$ and $V \cap W = \emptyset$. This implies that $Cl(V) \cap W = \emptyset$. Now $Cl(V) \cap U^c \subset Cl(V) \cap W = \emptyset$ and so $Cl(V) \subset U$.

(ii) $\Rightarrow$(iii). Let A and B be disjoint $\gamma_{g\alpha g}$-closed sets of (X,τ). Since $A \cap B = \emptyset$, $A \subset B^c$ and B^c is $\gamma_{g\alpha g}$-open. By assumption, there exists an open set U containing A such that $Cl(U) \subset B^c$ and so $Cl(U) \cap B = \emptyset$.

(iii) $\Rightarrow$(iv). Let A and B be disjoint $\gamma_{g\alpha g}$-closed sets of (X,τ). Then by assumption, there exists an open set U containing A such that $Cl(U) \cap B = \emptyset$. Since Cl(U) is closed, it is $\gamma_{g\alpha g}$-closed and so Cl(U) and B are an open set $\gamma_{g\alpha g}$-closed sets in X. Therefore again by assumption, there exist an open set containing B such that $Cl(U) \cap Cl(V) = \emptyset$.

(iv)$\Rightarrow$(i). Let A and B be any two disjoint $\gamma_{g\alpha g}$-closed sets of (X,τ). By assumption, there exists open sets U containing A and V containing B such that $Cl(U) \cap Cl(V) = \emptyset$. So we have $A \cap B = \emptyset$. Hence (X,τ) is $\gamma_{g\alpha g}$-normal.

THEOREM 6.2.10: A topological space (X,τ) is $\gamma_{g\alpha g}$-normal if and only if for any disjoint $\gamma_{g\alpha g}$-closed sets A and B of X there exists open sets U and V such that $A\subset U$, $B\subset V$ & $Cl(U)\cap Cl(V)=\varnothing$.

PROOF: Follows from the theorem 6.2.9.

THEOREM 6.2.11: Let (X,τ) be a $\gamma_{g\alpha g}$-normal space and Y be a subspace of X. Then the subspace Y is $\gamma_{g\alpha g}$-normal.

PROOF: Let A and B be any disjoint $\gamma_{g\alpha g}$-closed sets of Y. Then A and B are $\gamma_{g\alpha g}$-closed sets in (X,τ). Since X is $\gamma_{g\alpha g}$-normal, there exists disjoint open sets U & V of X such that $A\subset U$ & $B\subset V$. So $U\cap Y$ and $V\cap Y$ are disjoint open subsets of the subspace Y such that $A\subset U\cap Y$ and $B\subset V\cap Y$. This shows that the subspace Y is $\gamma_{g\alpha g}$-normal.

COROLLARY 6.2.12: The property of being $\gamma_{g\alpha g}$-normal is closed is hereditary.

THEOREM 6.2.13: If $f:(X,\tau)\rightarrow(Y,\sigma)$ is $\gamma_{g\alpha g}$-irresolute, $\gamma_{g\alpha g}$-closed, continuous, injective and (Y,σ) is $\gamma_{g\alpha g}$-normal then (X,τ) is $\gamma_{g\alpha g}$-normal.

PROOF: Let A and B be disjoint $\gamma_{g\alpha g}$-closed sets of X. Since f is irresolute, $\gamma_{g\alpha g}$-closed, by theorem 5.2.6, f(A) and f(B) are $\gamma_{g\alpha g}$-closed sets of (Y,σ). Since f is injective f(A) and f(B) disjoint $\gamma_{g\alpha g}$-closed sets of (Y,σ). Since (Y,σ) is $\gamma_{g\alpha g}$-normal, then there exists disjoint open sets U and V such that $f(A) \subset U$ and $f(B) \subset V$. Thus, we obtain $A \subset f^{-1}(U)$, $B \subset f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \varnothing$. Since f is continuous, $f^{-1}(U)$ & $f^{-1}(V)$ are open in $(X,\tau) \Rightarrow (X,\tau)$ is $\gamma_{g\alpha g}$-normal.

We now introduce $\gamma_{g\alpha g}$ –regular spaces using $\gamma_{g\alpha g}$ –closed sets in topological spaces. Some characterizations of $\gamma_{g\alpha g}$ –regular spaces are obtained.

DEFINITON 6.2.14: A topological space (X,τ) is said to be **$\gamma_{g\alpha g}$-regular** if for each $\gamma_{g\alpha g}$-closed set F of X and each point $x \notin F$, there exists disjoint αg-open sets U & V of X such that $x \in U$ and $F \subset V$.

EXAMPLE 6.2.15: Let X ={a,b,c}, τ ={∅,{a},{b,c},X}. αgO(X)={∅, {a},{b},{c},{a,b},{a,c},{b,c},X}. $\gamma_{g\alpha g}$C(X)={∅,{a},{b},{c},{a,b},{a,c}, {b,c},X}. Then X is a $\gamma_{g\alpha g}$-regular space.

THEOREM 6.2.16: A topological space (X,τ) is said to be $\gamma_{g\alpha g}$-regular if (X,τ) is regular.

PROOF: Let (X,τ) be a $\gamma_{g\alpha g}$–regular. Then for each $\gamma_{g\alpha g}$–closed set F of X and each point $x \notin F$, there exists disjoint αg-open sets U & V of X such that $x \in U$ and $F \subset V$. Since each $\gamma_{g\alpha g}$–closed set is closed set and each αg –open set is an open set in X, we get for each closed set F of X and each point $x \notin F$, there exists disjoint open sets U & V of X such that $x \in U$ and $F \subset V \Rightarrow (X,\tau)$ is regular.

THEOREM 6.2.17: The following are equivalent in a topological space (X,τ)

(i) (X,τ) is $\gamma_{g\alpha g}$–regular space

(ii) For each $x \in X$ and each $\gamma_{g\alpha g}$–nhd A of x, there exists an αg-nhd V of x such that $C_\alpha(V) \subset A$.

PROOF: (i)⇒(ii). Let A be any $\gamma_{g\alpha g}$–nhd A of x. Then there exists an $\gamma_{g\alpha g}$–open set G such that $x \in G \subset A$. Since X-G is $\gamma_{g\alpha g}$–closed and $x \notin X\text{-}G$, by hypothesis, there exist αg-open sets U & V such that $X\text{-}G \subset U$, $x \in V$ and $U \cap V = \emptyset$. So $V \subset X\text{-}U$. Now $C_\alpha(V) \subset C_\alpha(X\text{-}U) = X\text{-}U$ and $X\text{-}G \subset U \Rightarrow X\text{-}U \subset G \subset A$. Hence $C_\alpha(V) \subset A$.

(ii)⇒(i). Let F be any $\gamma_{g\alpha g}$–closed set of X and $x \notin F$. Then $x \in X\text{-}F$ and X-F is $\gamma_{g\alpha g}$–open and X-F is an $\gamma_{g\alpha g}$–nhd of x. By hypothesis, there exists an αg-nhd V of x such that $x \in V$ and $C_\alpha(V) \subset X\text{-}F \Rightarrow F \subset X\text{-} C_\alpha(V)$. Then

X- $C_\alpha(V)$ is an αg-open set containing F and $V\cap(X- C_\alpha(V)) = \varnothing$. So (X,τ) is $\gamma_{g\alpha g}$–regular.

DEFINITION 6.2.18: A space (X,τ) is said to be **$\gamma_{g\alpha g}$–symmetric** if for all $x,y\in X$, $x\neq y$, $x\in\gamma_{g\alpha g}$-Cl({y}) and $y\in \gamma_{g\alpha g}$-Cl({x}).

THEOREM 6.2.19: A topological space (X, τ) is $\gamma_{g\alpha g}$–symmetric if and only if {x} is $\gamma_{g\alpha g}$-closed for each $x\in X$.

PROOF: Assume that $x\in \gamma_{g\alpha g}$-Cl({y}) but $y\notin \gamma_{g\alpha g}$-Cl({x}). This means that $\gamma_{g\alpha g}$-Cl$(\{x\})^c$ contains y. This implies that $\gamma_{g\alpha g}$-Cl({y}) is a subset of $\gamma_{g\alpha g}$-Cl$(\{x\})^c$. But then $\gamma_{g\alpha g}$-Cl$(\{x\})^c$ contains x which is a contradiction. Conversely, suppose that $\{x\}\subset U$, where U is $\gamma_{g\alpha g}$-open set but $\gamma_{g\alpha g}$-Cl({x}) is not a subset of U. This means that $\gamma_{g\alpha g}$-Cl({x}) and U^c are not disjoint. Let y belongs to their intersection. Now we have $x\in \gamma_{g\alpha g}$-Cl({y}) which is a subset of U^c and $x\notin U$. But this is a contradiction to our assumption. Hence the result.

COROLLARY 6.2.20: If a topological space (X, τ) is $\gamma_{g\alpha g}$-T_1 space then it is $\gamma_{g\alpha g}$- symmetric.

PROOF: In a $\gamma_{g\alpha g}$-T_1 space, singleton sets are $\gamma_{g\alpha g}$- closed and therefore αg-closed. So by Theorem 6.2.19, the space is α-symmetric.

COROLLARY 6.2.21: For a topological space (X, τ) the following are equivalent

(i) (X, τ) is $\gamma_{g\alpha g}$- symmetric and $\gamma_{g\alpha g}$-T_0

(ii)(X, τ) is $\gamma_{g\alpha g}$-T_1.

PROOF: (i) $\Rightarrow$(ii): Let $x\neq y$ and by $\gamma_{g\alpha g}$-T_0, we may assume that $x\in G_1\subset\{y\}^c$ for $G_1\in\gamma_{g\alpha g}O(X,\tau)$. Then $x\notin \gamma_{g\alpha g}$-$C_\alpha(\{y\})$ and $y\notin \gamma_{g\alpha g}-C_\alpha(\{x\})$. There exists a $G_2\in\gamma_{g\alpha g}O(X,\tau)$ such that $y\in G_2\subset\{x\}^c$ and (X, τ) is $\gamma_{g\alpha g}$- T_1 space.

(ii) $\Rightarrow$(i) By corollary 6.2.20, (X, τ) is $\gamma_{g\alpha g}$- symmetric. Also it is trivial that (X, τ) is $\gamma_{g\alpha g}$- $T_1\Rightarrow$(X, τ) is $\gamma_{g\alpha g}$- T_0.

THEOREM 6.2.22: Every α-normal, $\gamma_{g\alpha g}$-symmetric space (X,τ) is α-regular.

PROOF: Let F be a closed subset of (X,τ) and x∈X such that x∉F. Since (X,τ) is $\gamma_{g\alpha g}$-symmetric space, by theorem 6.2.20, {x} is $\gamma_{g\alpha g}$-closed. Since F is closed and (X,τ) is α-normal, there exist disjoint α-open sets U and V such that F⊂U and {x}⊂V. So (X,τ) is α-regular.

THEOREM 6.2.23:A topological space (X,τ) is $\gamma_{g\alpha g}$-regular if and only for each $\gamma_{g\alpha g}$-closed set F of X and each point x∉F, there exist open sets U & V of X such that x∈U of X such that x∈U and F⊂V and Cl(U)∩Cl(V) = ∅.

PROOF: (Necessity): Let F be an $\gamma_{g\alpha g}$-closed set of X and x∉F. Then there exist open sets U_o and V of X such that x∈U_o, F⊂V and U_o∩V=∅, hence U_o∩Cl(V)=∅. Since (X,τ) is $\gamma_{g\alpha g}$-regular, there exist open sets U = U_o∩G, then U and V are open sets of X such that x∈U, F⊂V and Cl(U)∩Cl(V) =∅.

(Sufficiency): Sufficiency is obvious.

THEOREM 6.2.24: If (X,τ) is an $\gamma_{g\alpha g}$-regular space. Let Y be a subspace of X and $\gamma_{g\alpha g}$-closed subset of (X,τ) then Y is $\gamma_{g\alpha g}$-regular.

PROOF: Let A be any $\gamma_{g\alpha g}$-closed subset of Y and y∉A. Then A is $\gamma_{g\alpha g}$-closed set in (X,τ). Since (X,τ) is $\gamma_{g\alpha g}$-regular, there exist disjoint open sets U and V of X such that y∈U and A⊂V. Therefore U∩Y and V∩Y are disjoint open sets of the subspace Y such that y∈U∩Y and A⊂V∩Y. This shows that the subspace Y is $\gamma_{g\alpha g}$-regular.

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